A CLASS OF C^*-ALGEBRAS GENERALIZING BOTH GRAPH ALGEBRAS AND HOMEOMORPHISM C^*-ALGEBRAS II, EXAMPLES

TAKESHI KATSURA

Abstract. We show that the method to construct C^*-algebras from topological graphs, introduced in our previous paper, generalizes many known constructions. We give many ways to make new topological graphs from old ones, and study the relation of C^*-algebras constructed from them. We also give a characterization of our C^*-algebras in terms of their representation theory.

0. Introduction

In a previous paper [K1], we introduced the notion of topological graphs and a method to construct C^*-algebras from them. Topological graphs generalize ordinary graphs and homeomorphisms on locally compact spaces, and our method for constructing C^*-algebras from them generalizes the constructions of graph algebras and homeomorphism algebras (see [K1] for detail). In this paper, we give many ways to make new topological graphs from old ones, and study C^*-algebras constructed from them. We also see that the way of constructing C^*-algebras from topological graphs and the class of such C^*-algebras relates many known constructions and classes besides graph algebras and homeomorphism algebras. In [K1], we show that our C^*-algebras are always nuclear and satisfy the Universal Coefficient Theorem. So far, we know of no examples which satisfy these two conditions, but are not in our class. Almost all "classifiable" C^*-algebras can be obtained as C^*-algebras of topological graphs. Thus our C^*-algebras are useful to study the structure of classifiable C^*-algebras.

In Section 1, we recall definitions and results in our previous paper [K1]. In Section 2, we define factor maps between two topological graphs and show that these give $*$-homomorphisms between C^*-algebras associated with them. In Section 3, we investigate C^*-algebras $C^*(T)$ generated by Toeplitz pairs $T = (T^0, T^1)$. Thanks to this investigation, we get a characterization of our C^*-algebra $\mathcal{O}(E)$ without using the space E^0_{rg} (Proposition 3.23). We use the results here in the next paper [K5]. In Section 4, we define projective systems of topological graphs and their projective limits, and study how these relate to C^*-algebras $\mathcal{T}(E)$ and $\mathcal{O}(E)$. In Section 5 and Section 6, we give methods to make a new topological graph from given one so that the C^*-algebras they define are strongly Morita equivalent. Section 7 is devoted to give other operations to make new topological graphs. In the final three sections,

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we discuss examples which show how a number of C^*-algebras that appear in the literature may be realized as topological graph algebras.

1. Preliminaries

Definition 1.1. A *topological graph* $E = (E^0, E^1, d, r)$ consists of two locally compact spaces E^0 and E^1, and two maps $d, r\colon E^1 \to E^0$, where d is locally homeomorphic and r is continuous.

We regard an element v of E^0 as a vertex, and an element e of E^1 as a directed edge pointing from its domain $d(e) \in E^0$ to its range $r(e) \in E^0$. For a topological graph $E = (E^0, E^1, d, r)$, the triple (E^1, d, r) is called a *topological correspondence* on E^0, which is considered as a generalization of a continuous map. By the local homeomorphism $d\colon E^1 \to E^0$, E^1 is "locally" isomorphic to E^0, and the pair (E^1, d) defines a "domain" of a continuous map r which is "locally" a continuous map from E^0 to E^0.

Let us denote by $C_d(E^1)$ the set of continuous functions ξ on E^1 such that $\langle \xi, \xi \rangle(v) = \sum_{e \in d^{-1}(v)} |\xi(e)|^2 < \infty$ for any $v \in E^0$ and $\langle \xi, \xi \rangle \in C_0(E^0)$. For $\xi, \eta \in C_d(E^1)$ and $f \in C_0(E^0)$, we define $\xi f \in C_d(E^1)$ and $\langle \xi, \eta \rangle \in C_0(E^0)$ by

$$(\xi f)(e) = \xi(e) f(d(e)) \text{ for } e \in E^1$$

$$\langle \xi, \eta \rangle(v) = \sum_{e \in d^{-1}(v)} \overline{\xi(e)} \eta(e) \text{ for } v \in E^0.$$

With these operations, $C_d(E^1)$ is a (right) Hilbert $C_0(E^0)$-module ([K1, Proposition 1.10]). We define a map $\pi\colon C_b(E^1) \to \mathcal{L}(C_d(E^1))$ by $(\pi(f)\xi)(e) = f(e)\xi(e)$ for $f \in C_b(E^1)$, $\xi \in C_d(E^1)$ and $e \in E^1$, where $C_b(E^1)$ is the set of all bounded continuous functions on E^1. We have $\pi(f) \in \mathcal{K}(C_d(E^1))$ if and only if $f \in C_0(E^1)$ ([K1, Proposition 1.17]). We define a left action π_r of $C_0(E^0)$ on $C_d(E^1)$ by $\pi_r(f) = \pi(f \circ r) \in \mathcal{L}(C_d(E^1))$ for $f \in C_0(E^0)$. Thus we get a C^*-correspondence $C_d(E^1)$ over $C_0(E^0)$.

We set $d^0 = r^0 = \mathrm{id}_{E^0}$ and $d^1 = d, r^1 = r$. For $n = 2, 3, \ldots$, we recursively define a space E^n of paths with length n and domain and range maps $d^n, r^n\colon E^n \to E^0$ by

$$E^n = \{(e', e) \in E^1 \times E^{n-1} \mid d^1(e') = r^{n-1}(e)\},$$

$d^n((e', e)) = d^{n-1}(e)$ and $r^n((e', e)) = r^1(e')$. For each $n \in \mathbb{N}$, d^n is a local homeomorphism of E^n to E^0 and of course r^n is continuous (the triple (E^n, d^n, r^n) is the n-times composition of the topological correspondence (E^1, d, r) on E^0, see [K1, Section 1]). Thus we may define a C^*-correspondence $C_{d^n}(E^n)$ over $C_0(E^0)$ in a fashion similar to the definition of $C_d(E^1)$. We have that $C_{d^{n+m}}(E^{n+m}) \cong C_{d^n}(E^n) \otimes C_{d^m}(E^m)$ as C^*-correspondences over $C_0(E^0)$ for any $n, m \in \mathbb{N} = \{0, 1, 2, \ldots\}$. As long as no confusion arises, we omit the superscript n and simply write d, r for d^n, r^n.

Definition 1.2. Let $E = (E^0, E^1, d, r)$ be a topological graph. A *Toeplitz E-pair* in a C^*-algebra A is a pair of maps $T = (T^0, T^1)$ consisting of a $*$-homomorphism $T^0\colon C_0(E^0) \to A$ and a linear map $T^1\colon C_d(E^1) \to A$ satisfying

(i) $T^1(\xi)^* T^1(\eta) = T^0(\langle \xi, \eta \rangle)$ for $\xi, \eta \in C_d(E^1)$,
(ii) $T^0(f) T^1(\xi) = T^1(\pi_r(f)\xi)$ for $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$.

We denote by $\mathcal{T}(E)$ the universal C^*-algebra generated by a Toeplitz E-pair.

For a Toeplitz E-pair $T=(T^0,T^1)$, the equation $T^1(\xi)T^0(f)=T^1(\xi f)$ holds automatically from the condition (i). We write $C^*(T)$ to denote the C^*-algebra generated by the images of the maps T^0 and T^1. We define a $*$-homomorphism $\Phi\colon \mathcal{K}(C_d(E^1))\to C^*(T)$ by $\Phi(\theta_{\xi,\eta})=T^1(\xi)T^1(\eta)^*$ for $\xi,\eta\in C_d(E^1)$. We say that a Toeplitz E-pair $T=(T^0,T^1)$ is *injective* if T^0 is injective. If a Toeplitz E-pair $T=(T^0,T^1)$ is injective, then T^1 and Φ are isometric.

Definition 1.3. Let $E=(E^0,E^1,d,r)$ be a topological graph. We define three open subsets $E^0_{\mathrm{sce}},E^0_{\mathrm{fin}}$ and E^0_{rg} of E^0 by $E^0_{\mathrm{sce}}=E^0\setminus\overline{r(E^1)}$,

$$E^0_{\mathrm{fin}}=\{v\in E^0 \mid \text{ there exists a neighborhood } V \text{ of } v \\ \text{ such that } r^{-1}(V)\subset E^1 \text{ is compact}\},$$

and $E^0_{\mathrm{rg}}=E^0_{\mathrm{fin}}\setminus\overline{E^0_{\mathrm{sce}}}$. We define two closed subsets E^0_{inf} and E^0_{sg} of E^0 by $E^0_{\mathrm{inf}}=E^0\setminus E^0_{\mathrm{fin}}$ and $E^0_{\mathrm{sg}}=E^0\setminus E^0_{\mathrm{rg}}$.

We have $E^0_{\mathrm{sg}}=E^0_{\mathrm{inf}}\cup\overline{E^0_{\mathrm{sce}}}$. A vertex in E^0_{sce} is called a *source*. Vertices in E^0_{rg} are said to be *regular*, and those in E^0_{sg} are said to be *singular*. The following is a characterization of regular vertices.

Lemma 1.4 ([K1, Proposition 2.8]). *For $v\in E^0$, we have $v\in E^0_{\mathrm{rg}}$ if and only if there exists a neighborhood V of v such that $r^{-1}(V)$ is compact and $r(r^{-1}(V))=V$.*

Note that if it exists, such a neighborhood V is compact, and every compact neighborhood V' of v contained in V satisfies the same conditions. We have that $\ker\pi_r=C_0(E^0_{\mathrm{sce}})$ and $\pi_r^{-1}(\mathcal{K}(C_d(E^1)))=C_0(E^0_{\mathrm{fin}})$ ([K1, Proposition 1.24]). Hence the restriction of π_r to $C_0(E^0_{\mathrm{rg}})$ is an injection into $\mathcal{K}(C_d(E^1))$.

Definition 1.5. A Toeplitz E-pair $T=(T^0,T^1)$ is called a *Cuntz-Krieger E-pair* if $T^0(f)=\Phi(\pi_r(f))$ for any $f\in C_0(E^0_{\mathrm{rg}})$.

The universal C^*-algebra generated by a Cuntz-Krieger E-pair $t=(t^0,t^1)$ is denoted by $\mathcal{O}(E)$.

In [K2], the author suggests the way to associate a C^*-algebra $\mathcal{O}_X$ for each C^*-correspondence X, which is a modification of the construction of Pimsner algebras in [P]. The C^*-algebra $\mathcal{O}(E)$ is nothing but the C^*-algebra $\mathcal{O}_{C_d(E^1)}$ associated with the C^*-correspondence $C_d(E^1)$.

Since $t=(t^0,t^1)$ is injective ([K1, Proposition 3.7]), $\varphi\colon\mathcal{K}(C_d(E^1))\to\mathcal{O}(E)$ is injective. By the universality of $\mathcal{O}(E)$, there exists an action $\beta\colon\mathbb{T}\curvearrowright\mathcal{O}(E)$ defined by $\beta_z(t^0(f))=t^0(f)$ and $\beta_z(t^1(\xi))=zt^1(\xi)$ for $f\in C_0(E^0)$, $\xi\in C_d(E^1)$ and $z\in\mathbb{T}$. The action β is called the *gauge action*. We say that a Toeplitz E-pair T *admits a gauge action* if there exists an automorphism β'_z on $C^*(T)$ with $\beta'_z(T^0(f))=T^0(f)$ and $\beta'_z(T^1(\xi))=zT^1(\xi)$ for every $z\in\mathbb{T}$. Note that if such automorphisms β'_z exist, then $\beta'\colon\mathbb{T}\ni z\mapsto\beta'_z\in\mathrm{Aut}(C^*(T))$ becomes automatically a strongly continuous homomorphism. The following proposition is called the gauge-invariant uniqueness theorem.

Proposition 1.6 ([K1, Theorem 4.5]). *For a topological graph E and a Cuntz-Krieger E-pair T, the natural surjection $\mathcal{O}(E)\to C^*(T)$ is an isomorphism if and only if T is injective and admits a gauge action.*

2. Factor maps

In this section, we define factor maps between two topological graphs and show that these give $*$-homomorphisms between C^*-algebras associated with them. Let us take two topological graphs $E = (E^0, E^1, d_E, r_E)$ and $F = (F^0, F^1, d_F, r_F)$, and fix them. For a locally compact space X, we denote by $\widetilde{X} = X \cup \{\infty\}$ the one-point compactification of X. We consider elements of $C_0(X)$ as continuous functions on $\widetilde{X}$ vanishing at $\infty \in \widetilde{X}$.

Definition 2.1. A *factor map* from F to E is a pair $m = (m^0, m^1)$ consisting of continuous maps $m^0\colon \widetilde{F}^0 \to \widetilde{E}^0$ and $m^1\colon \widetilde{F}^1 \to \widetilde{E}^1$ which send ∞ to ∞, such that

(i) For every $e \in F^1$ with $m^1(e) \in E^1$, we have $r_E(m^1(e)) = m^0(r_F(e))$ and $d_E(m^1(e)) = m^0(d_F(e))$.
(ii) If $e' \in E^1$ and $v \in F^0$ satisfies $d_E(e') = m^0(v)$, then there exists a unique element $e \in F^1$ such that $m^1(e) = e'$ and $d_F(e) = v$.

The term 'factor map' comes from cosidering topological graphs as dynamical systems. Note that the domain and range maps $d_E, r_E\colon E^1 \to E^0$ of a topological graph $E = (E^0, E^1, d_E, r_E)$ may not extend to continuous maps from $\widetilde{E}^1$ to $\widetilde{E}^0$ in general.

Lemma 2.2. *Let $m = (m^0, m^1)$ be a factor map from F to E. Then m induces a pair of maps $\mu = (\mu^0, \mu^1)$ by*

$$\mu^0\colon C_0(E^0) \ni f \mapsto f \circ m^0 \in C_0(F^0)$$
$$\mu^1\colon C_{d_E}(E^1) \ni \xi \mapsto \xi \circ m^1 \in C_{d_F}(F^1),$$

which is a morphism in the sense of [K3, Definition 2.3]*, that is, μ^0 is a $*$-homomorphism, $\langle \mu^1(\xi), \mu^1(\eta)\rangle = \mu^0(\langle \xi, \eta\rangle)$ and $\pi_{r_F}(\mu^0(f))\mu^1(\xi) = \mu^1(\pi_{r_E}(f)\xi)$ for $\xi, \eta \in C_{d_E}(E^1)$ and $f \in C_0(E^0)$.*

Proof. Clearly $f \mapsto f \circ m^0$ defines a $*$-homomorphism μ^0. Take $\xi, \eta \in C_{d_E}(E^1)$, and we will show the equality $\langle \mu^1(\xi), \mu^1(\eta)\rangle = \mu^0(\langle \xi, \eta\rangle)$. Let us take $v \in F^0$. For $e \in (d_F)^{-1}(v) \cap (m^1)^{-1}(E^1)$, we have $m^1(e) \in (d_E)^{-1}(m^0(v))$ by the condition (i) in Definition 2.1. Conversely for $e' \in (d_E)^{-1}(m^0(v))$, there exists unique $e \in (d_F)^{-1}(v)$ with $m^1(e) = e'$. Hence the map

$$(d_F)^{-1}(v) \cap (m^1)^{-1}(E^1) \ni e \mapsto m^1(e) \in (d_E)^{-1}(m^0(v))$$

is bijective. Therefore we have

$$\begin{aligned}
\langle \mu^1(\xi), \mu^1(\eta)\rangle(v) &= \sum_{e \in (d_F)^{-1}(v)} \overline{\mu^1(\xi)(e)}\, \mu^1(\eta)(e) \\
&= \sum_{e \in (d_F)^{-1}(v) \cap (m^1)^{-1}(E^1)} \overline{\xi(m^1(e))}\, \eta(m^1(e)) \\
&= \sum_{e' \in (d_E)^{-1}(m^0(v))} \overline{\xi(e')}\eta(e') \\
&= \langle \xi, \eta\rangle(m^0(v)) \\
&= \mu^0(\langle \xi, \eta\rangle)(v)
\end{aligned}$$

(note that when $m^0(v) = \infty$ we have $(d_E)^{-1}(m^0(v)) = \emptyset$, hence in this case the both hands of the fourth equality are zero). Thus we get $\langle \mu^1(\xi), \mu^1(\eta)\rangle = \mu^0(\langle \xi, \eta\rangle)$. By taking $\xi = \eta$ in the above equality, we see that μ^1 is well-defined. Finally it is easy to see $\mu^1(\pi_{r_E}(f)\xi) = \pi_{r_F}(\mu^0(f))\mu^1(\xi)$ for $f \in C_0(E^0)$ and $\xi \in C_{d_E}(E^1)$. We are done. □

By this lemma, we get the following proposition.

Proposition 2.3. *Let $\mu^0 \colon C_0(E^0) \to C_0(F^0)$, $\mu^1 \colon C_{d_E}(E^1) \to C_{d_F}(F^1)$ be maps defined from a factor map $m = (m^0, m^1)$ from F to E as above. Then there exists a unique $*$-homomorphism $\mu \colon \mathcal{T}(E) \to \mathcal{T}(F)$ such that $\mu \circ T_E^i = T_F^i \circ \mu^i$ for $i = 0, 1$, where $T_E = (T_E^0, T_E^1)$ and $T_F = (T_F^0, T_F^1)$ are the universal Toeplitz E-pair in $\mathcal{T}(E)$ and the universal Toeplitz F-pair in $\mathcal{T}(F)$, respectively.*

Proposition 2.4. *Let E, F, G be topological graphs, and m, n be factor maps from F to E and G to F respectively. We set $(m \circ n)^i = m^i \circ n^i \colon \widetilde{G}^i \to \widetilde{E}^i$ for $i = 0, 1$. Then $m \circ n = ((m \circ n)^0, (m \circ n)^1)$ is a factor map from G to E, and the $*$-homomorphism $\omega \colon \mathcal{T}(E) \to \mathcal{T}(G)$ defined from $m \circ n$ is the composition of $\mu \colon \mathcal{T}(E) \to \mathcal{T}(F)$ and $\nu \colon \mathcal{T}(F) \to \mathcal{T}(G)$ which are defined from m and n respectively.*

Proof. Take $e \in G^1$ with $(m^1 \circ n^1)(e) \in E^1$. Then $n^1(e) \in F^1$. Hence we have $d_F(n^1(e)) = n^0(d_G(e))$ and $r_F(n^1(e)) = n^0(r_G(e))$. Since $m^1(n^1(e)) \in E^1$, we have $d_E(m^1(n^1(e))) = m^0(d_F(n^1(e)))$ and $r_E(m^1(n^1(e))) = m^0(r_F(n^1(e)))$. Therefore we get

$$d_E((m^1 \circ n^1)(e)) = (m^0 \circ n^0)(d_G(e)), \quad r_E((m^1 \circ n^1)(e)) = (m^0 \circ n^0)(r_G(e)).$$

Take $e \in E^1$ and $v \in G^0$ with $d_E(e) = (m^0 \circ n^0)(v)$. Since m is a factor map, there exists unique $e' \in F^1$ with $m^1(e') = e$ and $d_F(e') = n^0(v)$. Since n is a factor map, there exists unique $e'' \in G^1$ with $n^1(e'') = e'$ and $d_G(e'') = v$. Therefore $e'' \in G^1$ is the unique element satisfying $m^1(n^1(e'')) = e$ and $d_G(e'') = v$. Thus $m \circ n$ is a factor map. By Proposition 2.3, we have

$$\omega \circ T_E^i = T_G^i \circ \omega^i = T_G^i \circ \nu^i \circ \mu^i = \nu \circ T_F^i \circ \mu^i = \nu \circ \mu \circ T_E^i$$

for $i = 0, 1$. Since $\mathcal{T}(E)$ is generated by the images of T_E^0 and T_E^1, we have $\omega = \nu \circ \mu$. □

The factor map $m \circ n$ defined in Proposition 2.4 is called the *composition* of factor maps m and n. Thus we get a contravariant functor $E \mapsto \mathcal{T}(E)$ from the category of topological graphs with factor maps as morphisms to the one of C^*-algebras with $*$-homomorphisms as morphisms. We study which factor maps from F to E give $*$-homomorphisms from $\mathcal{O}(E)$ to $\mathcal{O}(F)$. For a factor map $m = (m^0, m^1)$ from F to E, we can define a $*$-homomorphism $\psi \colon \mathcal{K}(C_{d_E}(E^1)) \to \mathcal{K}(C_{d_F}(F^1))$ by $\psi(\theta_{\xi,\eta}) = \theta_{\mu^1(\xi),\mu^1(\eta)}$ for $\xi, \eta \in C_{d_E}(E^1)$ where $\mu^1 \colon C_{d_E}(E^1) \to C_{d_F}(F^1)$ is defined from m^1 as above. To get a $*$-homomorphism from $\mathcal{O}(E)$ to $\mathcal{O}(F)$, we need to know whether we have $\psi(\pi_{r_E}(f)) = \pi_{r_F}(\mu^0(f))$ for $f \in C_0(E^0_{\mathrm{rg}})$. As defined in Section 1, the map $\pi_{r_E} \colon C_0(E^0) \to \mathcal{L}(C_d(E^1))$ is the composition of the map $C_0(E^0) \ni f \mapsto f \circ r_E \in C_b(E^1)$ and $\pi_E \colon C_b(E^1) \to \mathcal{L}(C_{d_E}(E^1))$. The map $\pi_{r_F} \colon C_0(F^0) \to \mathcal{L}(C_d(F^1))$ is also the composition of the map $C_0(F^0) \ni f \mapsto f \circ r_F \in C_b(F^1)$ and $\pi_F \colon C_b(F^1) \to \mathcal{L}(C_{d_F}(F^1))$. In order to get the equality $\psi(\pi_{r_E}(f)) = \pi_{r_F}(\mu^0(f))$ for

$f \in C_0(E^0_{\mathrm{rg}})$, it suffices to see that $\psi(\pi_E(g)) = \pi_F(\mu^1(g))$ for $g = f \circ r_E \in C_0(E^1)$ and that $\mu^1(g) = \mu^0(f) \circ r_F$. The former equality is valid for arbitrary factor maps.

Proposition 2.5. *Let $\psi\colon \mathcal{K}(C_{d_E}(E^1)) \to \mathcal{K}(C_{d_F}(F^1))$ be the $*$-homomorphism defined from a $*$-homomorphism $\mu^1\colon C_{d_E}(E^1) \to C_{d_F}(F^1)$ as above. Then we have $\psi(\pi_E(g)) = \pi_F(\mu^1(g))$ for all $g \in C_0(E^1)$.*

Proof. Since $\psi \circ \pi_E$ and $\pi_F \circ \mu^1$ are continuous, it suffices to show the equality only for elements in $C_c(E^1)$. Take $g \in C_c(E^1)$. By [K1, Lemma 1.16], there exist $\xi_k, \eta_k \in C_c(E^1)$ for $k = 1, \ldots, m$ such that $g = \sum_{k=1}^m \xi_k\overline{\eta_k}$ and that $\xi_k(e)\,\overline{\eta_k(e')} = 0$ for any k and any $e, e' \in E^1$ with $d_E(e) = d_E(e')$ and $e \neq e'$. By [K1, Lemma 1.15], we have $\pi_E(g) = \sum_{k=1}^m \theta_{\xi_k,\eta_k}$. Thus we have $\psi(\pi_E(g)) = \sum_{k=1}^m \theta_{\mu^1(\xi_k),\mu^1(\eta_k)}$. We also have $\mu^1(g) = \sum_{k=1}^m \mu^1(\xi_k)\overline{\mu^1(\eta_k)}$. To prove $\pi_F(\mu^1(g)) = \sum_{k=1}^m \theta_{\mu^1(\xi_k),\mu^1(\eta_k)}$ by [K1, Lemma 1.15], we need to check that $\mu^1(\xi_k)(e)\,\overline{\mu^1(\eta_k)(e')} = 0$ for each k and $e, e' \in F^1$ with $d_F(e) = d_F(e')$ and $e \neq e'$. If either e or e' is not in $(m^1)^{-1}(E^1)$, then clearly $\mu^1(\xi_k)(e)\,\overline{\mu^1(\eta_k)(e')} = 0$. When both e and e' are in $(m^1)^{-1}(E^1)$, $d_F(e) = d_F(e')$ implies $d_E(m^1(e)) = d_E(m^1(e'))$ by the condition (i) in Definition 2.1, and $e \neq e'$ implies $m^1(e) \neq m^1(e')$ by the condition (ii). Hence

$$\mu^1(\xi_k)(e)\,\overline{\mu^1(\eta_k)(e')} = \xi_k(m^1(e))\,\overline{\eta_k(m^1(e'))} = 0.$$

Thus [K1, Lemma 1.15] implies

$$\pi_F(\mu^1(g)) = \sum_{k=1}^m \theta_{\mu^1(\xi_k),\mu^1(\eta_k)} = \psi(\pi_E(g)).$$

We are done. □

The equality $\mu^1(f \circ r_E) = \mu^0(f) \circ r_F$ for $f \in C_0(E^0_{\mathrm{rg}})$ is not true for a general factor map $m = (m^0, m^1)$. We need the following notion.

Definition 2.6. A factor map $m = (m^0, m^1)$ from F to E is called *regular* if $(r_F)^{-1}(v)$ is non-empty and contained in $(m^1)^{-1}(E^1)$ for every $v \in F^0$ with $m^0(v) \in E^0_{\mathrm{rg}}$.

Lemma 2.7. *For a regular factor map m from F to E, we have $(m^0)^{-1}(E^0_{\mathrm{rg}}) \subset F^0_{\mathrm{rg}}$.*

Proof. Let us take $v \in (m^0)^{-1}(E^0_{\mathrm{rg}}) \subset F^0$. Take a compact neighborhood V of $m^0(v) \in E^0_{\mathrm{rg}}$ such that $V \subset E^0_{\mathrm{rg}}$, and set $U = (r_E)^{-1}(V) \subset E^1$. Then U is compact and $r_E(U) = V$ (see Lemma 1.4). Set $V' = (m^0)^{-1}(V) \subset F^0$ and $U' = (r_F)^{-1}(V') \subset F^1$. Then V' is a neighborhood of v. By the regularity of m, we have $r_F(U') = V'$ and $U' \subset (m^1)^{-1}(E^1)$. The condition (i) of Definition 2.1 tells us that $U' = (m^1)^{-1}(U)$. Since U is compact and m^1 is proper, we see that U' is compact. Thus we have found a neighborhood V' of v such that $(r_F)^{-1}(V') = U'$ is compact and $r_F((r_F)^{-1}(V')) = r_F(U') = V'$. By Lemma 1.4, we see that $v \in F^0_{\mathrm{rg}}$. Thus we have $(m^0)^{-1}(E^0_{\mathrm{rg}}) \subset F^0_{\mathrm{rg}}$. □

Lemma 2.8. *Let $m = (m^0, m^1)$ be a regular factor map from F to E, and define a $*$-homomorphism $\mu^0\colon C_0(E^0) \to C_0(F^0)$, a linear map $\mu^1\colon C_{d_E}(E^1) \to C_{d_F}(F^1)$ and a $*$-homomorphism $\psi\colon \mathcal{K}(C_{d_E}(E^1)) \to \mathcal{K}(C_{d_F}(F^1))$ as before. Then we have $\psi(\pi_{r_E}(f)) = \pi_{r_F}(\mu^0(f))$ for $f \in C_0(E^0_{\mathrm{rg}})$.*

Proof. By Proposition 2.5, it suffices to show that $\mu^1(f \circ r_E) = \mu^0(f) \circ r_F$ for $f \in C_0(E^0_{\mathrm{rg}})$. For $e \in (m^1)^{-1}(E^1)$, we have

$$\mu^1(f \circ r_E)(e) = f(r_E(m^1(e))) = f(m^0(r_F(e))) = \mu^0(f)(r_F(e)).$$

For $e \notin (m^1)^{-1}(E^1)$, we have $\mu^1(f \circ r_E)(e) = 0$. If $r_F(e) \notin (m^0)^{-1}(E^0)$ then $\mu^0(f)(r_F(e)) = 0$. If $r_F(e) \in (m^0)^{-1}(E^0)$ then we have $m^0(r_F(e)) \notin E^0_{\mathrm{rg}}$ by the regularity of the factor map m. Hence in this case, $\mu^0(f)(r_F(e)) = f(m^0(r_F(e))) = 0$. Therefore we have $\mu^1(f \circ r_E) = \mu^0(f) \circ r_F$. We are done. $\square$

Proposition 2.9. *Let $\mu^0 \colon C_0(E^0) \to C_0(F^0)$ and $\mu^1 : C_{d_E}(E^1) \to C_{d_F}(F^1)$ be maps defined from a regular factor map m from F to E. Then there exists a unique $*$-homomorphism $\mu \colon \mathcal{O}(E) \to \mathcal{O}(F)$ such that $\mu \circ t^i_E = t^i_F \circ \mu^i$ for $i = 0, 1$.*

The $*$-homomorphism μ is injective if and only if m^0 is surjective.

Proof. To define a $*$-homomorphism $\mu \colon \mathcal{O}(E) \to \mathcal{O}(F)$ such that $\mu \circ t^i_E = t^i_F \circ \mu^i$ for $i = 0, 1$, it suffices to check that the pair of maps $T^0 = t^0_F \circ \mu^0 \colon C_0(E^0) \to \mathcal{O}(F)$ and $T^1 = t^1_F \circ \mu^1 \colon C_{d_E}(E^1) \to \mathcal{O}(F)$ is a Cuntz-Krieger E-pair. We already saw that $T = (T^0, T^1)$ is a Toeplitz E-pair in Proposition 2.3. The map $\Phi \colon \mathcal{K}(C_{d_E}(E^1)) \to \mathcal{O}(F)$ defined by $\Phi(\theta_{\xi,\eta}) = T^1(\xi)T^1(\eta)^*$ satisfies the equation $\Phi = \varphi_F \circ \psi$. For $f \in C_0(E^0_{\mathrm{rg}})$, we have $\mu^0(f) \in C_0(F^0_{\mathrm{rg}})$ by Lemma 2.7. Hence we have

$$T^0(f) = t^0_F(\mu^0(f)) = \varphi_F(\pi_{r_F}(\mu^0(f))) = \varphi_F(\psi(\pi_{r_E}(f))) = \Phi(\pi_{r_E}(f))$$

by Lemma 2.8. This implies that T is a Cuntz-Krieger E-pair. Therefore there exists a $*$-homomorphism $\mu \colon \mathcal{O}(E) \to \mathcal{O}(F)$ such that $\mu \circ t^i_E = t^i_F \circ \mu^i$ for $i = 0, 1$. The uniqueness is easily verified.

The C^*-algebra $\mathcal{O}(F)$ has the gauge action β and we see that $\beta_z(T^0(f)) = T^0(f)$ and $\beta_z(T^1(\xi)) = zT^1(\xi)$ for $f \in C_0(E^0)$, $\xi \in C_d(E^1)$ and $z \in \mathbb{T}$. Hence by Proposition 1.6 the $*$-homomorphism μ is injective if and only if $T^0 = t^0_F \circ \mu^0$ is injective. Since t^0_F is injective, $t^0_F \circ \mu^0$ is injective if and only if so is μ^0. It is easy to see that μ^0 is injective exactly when m^0 is surjective. Thus μ is injective if and only if m^0 is surjective. $\square$

Note that if m^0 is surjective then so is m^1 by the condition (ii) in Definition 2.1.

Proposition 2.10. *Let E, F, G be topological graphs, and m, n be regular factor maps from F to E and G to F respectively. Then the composition $m \circ n$ of m and n is regular and the $*$-homomorphism $\mathcal{O}(E) \to \mathcal{O}(G)$ defined from $m \circ n$ is the composition of the two maps $\mathcal{O}(E) \to \mathcal{O}(F)$ and $\mathcal{O}(F) \to \mathcal{O}(G)$ defined from m and n respectively.*

Proof. Take $v \in G^0$ with $(m^0 \circ n^0)(v) \in E^0_{\mathrm{rg}}$. We have $v \in G^0_{\mathrm{rg}}$ because Lemma 2.7 implies

$$(m^0 \circ n^0)^{-1}(E^0_{\mathrm{rg}}) \subset (n^0)^{-1}(F^0_{\mathrm{rg}}) \subset G^0_{\mathrm{rg}}.$$

Hence the set $(r_G)^{-1}(v)$ is not empty. Take $e \in (r_G)^{-1}(v)$. Since $n^0(r_G(e)) \in (m^0)^{-1}(E^0_{\mathrm{rg}}) \subset F^0_{\mathrm{rg}}$, we have $n^1(e) \in F^1$. Since $m^0\big(r_F(n^1(e))\big) = m^0(n^0(r_G(e))) \in E^0_{\mathrm{rg}}$, we have $m^1(n^1(e)) \in E^1$. Thus $m \circ n$ is regular. The remainder of the proof is similar to that of Proposition 2.4. $\square$

3. C^*-algebras generated by Toeplitz pairs

In this section, we investigate C^*-algebras $C^*(T)$ generated by Toeplitz pairs $T = (T^0, T^1)$. To this end, we introduce a construction of new topological graph E_Y from an original topological graph E and a closed subset Y of E^0_{rg}, and see that Cuntz-Krieger E_Y-pairs are useful to study Toeplitz E-pairs. We will use the rsults in this section for analysing ideal structures of $\mathcal{O}(E)$ in [K5].

Definition 3.1. Let E be a topological graph. For a Toeplitz E-pair $T = (T^0, T^1)$, we define a closed subset Y_T of E^0_{rg} by

$$C_0(E^0_{\mathrm{rg}} \setminus Y_T) = \{f \in C_0(E^0_{\mathrm{rg}}) \mid T^0(f) = \Phi(\pi_r(f))\}.$$

It is not difficult to see that the right hand side of the equation above is an ideal of $C_0(E^0_{\mathrm{rg}})$. We have $Y_T = \emptyset$ if and only if T is a Cuntz-Krieger E-pair. Thus Y_T measures how far T is from being a Cuntz-Krieger E-pair.

Lemma 3.2. *Let E be a topological graph and T be an injective Toeplitz E-pair. For $f \in C_0(E^0)$, we have $T^0(f) \in \Phi\big(\mathcal{K}(C_d(E^1))\big)$ if and only if $f \in C_0(E^0_{\mathrm{rg}} \setminus Y_T)$.*

Proof. For $f \in C_0(E^0_{\mathrm{rg}} \setminus Y_T)$, we have $T^0(f) = \Phi(\pi_r(f)) \in \Phi\big(\mathcal{K}(C_d(E^1))\big)$. Conversely take $f \in C_0(E^0)$ with $T^0(f) \in \Phi\big(\mathcal{K}(C_d(E^1))\big)$. By [K1, Proposition 2.11], we have $f \in C_0(E^0_{\mathrm{rg}})$ and $T^0(f) = \Phi(\pi_r(f))$. Hence we get $f \in C_0(E^0_{\mathrm{rg}} \setminus Y_T)$. □

We will construct a topological graph E_Y from a topological graph E and a closed subset Y of E^0_{rg}, and prove that there exists a one-to-one correspondence between Toeplitz E-pairs T with $Y_T \subset Y$ and Cuntz-Krieger E_Y-pairs. This enables us to use results on Cuntz-Krieger pairs for analyzing C^*-algebras $C^*(T)$ generated by the Toeplitz pairs T. This fact also gives a new definition of $\mathcal{O}(E)$ which does not use the space E^0_{rg} or the notion of Cuntz-Krieger pairs (Proposition 3.23).

Let us take a topological graph $E = (E^0, E^1, d, r)$ and a closed subset Y of E^0_{rg}. We define a topological graph $E_Y = (E^0_Y, E^1_Y, d_Y, r_Y)$ as follows. Set $\partial Y = \overline{Y} \setminus Y$ where $\overline{Y}$ is taken in E^0. Since Y is a closed subset of an open subset E^0_{rg}, we see that Y is open in $\overline{Y}$. Hence ∂Y is closed in $\overline{Y}$. A locally compact space $E^0_Y = E^0 \amalg_{\partial Y} \overline{Y}$ is defined to be a topological space obtained from the disjoint union $E^0 \amalg \overline{Y}$ by identifying the common closed subset ∂Y. Similarly we define $E^1_Y = E^1 \amalg_{d^{-1}(\partial Y)} d^{-1}(\overline{Y})$. Note that we have $d^{-1}(\overline{Y}) = \overline{d^{-1}(Y)}$ and $d^{-1}(\partial Y) = \partial(d^{-1}(Y))$ because $d\colon E^1 \to E^0$ is locally homeomorphic. We consider E^0 and E^1 as subsets of E^0_Y and E^1_Y, respectively. Both inclusions $Y \to \overline{Y} \subset E^0_Y$ and $d^{-1}(Y) \to d^{-1}(\overline{Y}) \subset E^1_Y$ are denoted by ω. Thus $\omega(Y)$ and $\omega(d^{-1}(Y))$ are the complements of the closed subsets $E^0 \subset E^0_Y$ and $E^1 \subset E^1_Y$, respectively. We may extend d and r to maps d_Y and r_Y on E^1_Y by setting

$$d_Y(\omega(e)) = \omega(d(e)) \in \omega(Y) \subset E^0_Y, \quad \text{and} \quad r_Y(\omega(e)) = r(e) \in E^0 \subset E^0_Y$$

for $e \in d^{-1}(Y)$. It is not difficult to see that $d_Y\colon E^1_Y \to E^0_Y$ is a local homeomorphism and $r_Y\colon E^1_Y \to E^0_Y$ is a continuous map. Thus we get a topological graph $E_Y = (E^0_Y, E^1_Y, d_Y, r_Y)$. Note that E_Y is obtained from the topological graph E by attaching extra vertices $\omega(Y)$ and extra edges $\omega(d^{-1}(Y))$ whose domains are in $\omega(Y)$ and ranges are in E^0.

Lemma 3.3. *We have $(E^0_Y)_{\mathrm{fin}} = E^0_{\mathrm{fin}} \cup \omega(Y)$, $(E^0_Y)_{\mathrm{sce}} = E^0_{\mathrm{sce}} \cup \omega(Y)$, and $(E^0_Y)_{\mathrm{rg}} = E^0_{\mathrm{rg}}$.*

Proof. For any $v \in Y$, we have $(r_Y)^{-1}(\omega(v)) = \emptyset$. Hence the open subset $\omega(Y)$ of E_Y^0 is contained in $(E_Y^0)_{sce}$. It is easy to see that for $v \in E^0 \setminus \partial Y$ we have $v \in (E_Y^0)_{fin}$ if and only if $v \in E^0_{\rm fin}$, and $v \in (E_Y^0)_{sce}$ if and only if $v \in E^0_{\rm sce}$. Let us take $v \in \partial Y$. For a neighborhood V of $v \in E^0$, the set $V' = V \cup \omega(V \cap Y) \subset E^0$ is a neighborhood of $v \in E_Y^0$, and we have $r_Y^{-1}(V') = r^{-1}(V)$. Since we can find a neighborhood of the above form in every neighborhood of $v \in \partial Y \subset E_Y^0$, we see that $v \in (E_Y^0)_{fin}$ if and only if $v \in E^0_{\rm fin}$, and that $v \in (E_Y^0)_{sce}$ if and only if $v \in E^0_{\rm sce}$. Therefore we get $(E_Y^0)_{fin} = E^0_{\rm fin} \cup \omega(Y)$ and $(E_Y^0)_{sce} = E^0_{\rm sce} \cup \omega(Y)$. Finally we have $(E_Y^0)_{\rm rg} = (E_Y^0)_{\rm fin} \setminus \overline{(E_Y^0)_{sce}} = E^0_{\rm rg} \setminus \partial Y$. Since Y is a closed subset of $E^0_{\rm rg}$, we have $E^0_{\rm rg} \cap \overline{Y} = Y$. Hence $E^0_{\rm rg} \cap \partial Y = \emptyset$. Thus we get $(E_Y^0)_{rg} = E^0_{\rm rg} \setminus \partial Y = E^0_{\rm rg}$. □

We define a map $m_Y^0 \colon E_Y^0 \to E^0$ and $m_Y^1 \colon E_Y^1 \to E^1$ by the identities on E^0 and E^1, and $m_Y^0(\omega(v)) = v$, $m_Y^1(\omega(e)) = e$ for $v \in V$ and $e \in d^{-1}(V)$. Both m_Y^0 and m_Y^1 are proper continuous surjections. Hence these extend the continuous maps $\widetilde{E}_Y^0 \to \widetilde{E}^0$ and $\widetilde{E}_Y^1 \to \widetilde{E}^1$, which are still denoted by m_Y^0 and m_Y^1. It is not difficult to see that the pair $m_Y = (m_Y^0, m_Y^1)$ is a factor map from E_Y to E. The factor map m_Y is not regular when $Y \neq \emptyset$ and m_Y is identity when $Y = \emptyset$. We define a $*$-homomorphism $\mu_Y^0 \colon C_0(E^0) \to C_0(E_Y^0)$ and a linear map $\mu_Y^1 \colon C_d(E^1) \to C_{d_Y}(E_Y^1)$ from $m_Y = (m_Y^0, m_Y^1)$. The $*$-homomorphism μ_Y^0 is an isomorphism onto the subalgebra

$$\{h \in C_0(E_Y^0) \mid h(v) = h(\omega(v)) \text{ for } v \in Y\},$$

and the linear map μ_Y^1 is an isometric map onto

$$\{\zeta \in C_{d_Y}(E_Y^1) \mid \zeta(e) = \zeta(\omega(e)) \text{ for } e \in d^{-1}(Y)\}.$$

Let $t_Y = (t_Y^0, t_Y^1)$ be the universal Cuntz-Krieger E_Y-pair on $\mathcal{O}(E_Y)$ and define a $*$-homomorphism $T_Y^0 \colon C_0(E^0) \to \mathcal{O}(E_Y)$ and a linear map $T_Y^1 \colon C_d(E^1) \to \mathcal{O}(E_Y)$ by $T_Y^i = t_Y^i \circ \mu_Y^i$ for $i = 0, 1$. Let $\varphi_Y \colon \mathcal{K}(C_{d_Y}(E_Y^1)) \to \mathcal{O}(E_Y)$ and $\Phi_Y \colon \mathcal{K}(C_d(E^1)) \to \mathcal{O}(E_Y)$ be $*$-homomorphisms determined by t_Y and T_Y, respectively. Note that we have $\Phi_Y = \varphi_Y \circ \psi$ where $\psi \colon \mathcal{K}(C_d(E^1)) \to \mathcal{K}(C_{d_Y}(E_Y^1))$ is defined by $\psi(\theta_{\xi,\eta}) = \theta_{\mu_Y^1(\xi), \mu_Y^1(\eta)}$ for $\xi, \eta \in C_d(E^1)$.

Lemma 3.4. *For an element $f \in C_0(E^0_{\rm rg})$, we have $\pi_{r_Y}(\mu^0(f)) = \psi(\pi_r(f)) \in \mathcal{K}(C_{d_Y}(E_Y^1))$.*

Proof. Take $f \in C_0(E^0_{\rm rg})$. By Proposition 2.5, it suffices to see that $\mu^0(f) \circ r_Y = \mu^1(f \circ r)$. This is clear because we have $m^0(r_Y(e)) = r(m^1(e))$ for all $e \in E^1$. The proof is completed. □

Proposition 3.5. *The pair $T_Y = (T_Y^0, T_Y^1)$ is an injective Toeplitz E-pair in $\mathcal{O}(E_Y)$ such that $Y_{T_Y} = Y$.*

Proof. By Proposition 2.3, T_Y is a Toeplitz E-pair. Clearly the pair T_Y is injective. We will show that $Y_{T_Y} = Y$. Take $f \in C_0(E^0_{\rm rg} \setminus Y)$. We have $\mu_Y^0(f) \in C_0((E_Y^0)_{\rm rg})$ by Lemma 3.3. We have

$$T_Y^0(f) = t_Y^0(\mu_Y^0(f)) = \varphi_Y\big(\pi_{r_Y}(\mu_Y^0(f))\big) = \varphi_Y\big(\psi(\pi_r(f))\big) = \Phi_Y(\pi_r(f))$$

by Lemma 3.4. Conversely take $f \in C_0(E^0_{\rm rg})$ with $T_Y^0(f) = \Phi_Y(\pi_r(f))$. We see

$$t_Y^0(\mu_Y^0(f)) = T_Y^0(f) = \Phi_Y(\pi_r(f)) \in \Phi_Y\big(\mathcal{K}(C_d(E^1))\big) \subset \varphi_Y\big(\mathcal{K}(C_d(E_Y^1))\big).$$

Hence we get $\mu_Y^0(f) \in C_0((E_Y^0)_{\mathrm{rg}})$ by Lemma 3.2. This implies that $f \in C_0(E^0_{\mathrm{rg}} \setminus Y)$ by Lemma 3.3. Thus we have

$$\{f \in C_0(E^0_{\mathrm{rg}}) \mid T_Y^0(f) = \Phi_Y(\pi_r(f))\} = C_0(E^0_{\mathrm{rg}} \setminus Y).$$

This means that $Y_{T_Y} = Y$. □

By Proposition 3.5, for any $*$-homomorphism $\rho\colon \mathcal{O}(E_Y) \to B$, the pair $T = (T^0, T^1)$ defined by $T^i = \rho \circ T_Y^i$ for $i = 0, 1$ is a Toeplitz E-pair in the C^*-algebra B satisfying $Y_T \subset Y$. We will prove the converse. Take a topological graph E, a closed subset Y of E^0_{rg} and a Toeplitz E-pair T with $Y_T \subset Y$. To get a $*$-homomorphism $\rho\colon \mathcal{O}(E_Y) \to C^*(T)$ such that $T^i = \rho \circ T_Y^i$ for $i = 0, 1$, it suffices to construct a Cuntz-Krieger E_Y-pair $\widetilde{T} = (\widetilde{T}^0, \widetilde{T}^1)$ on $C^*(T)$ such that $T^i = \widetilde{T}^i \circ \mu_Y^i$ for $i = 0, 1$.

Define a $*$-homomorphism $T^0_{\mathrm{rg}}\colon C_0(E^0_{\mathrm{rg}}) \to C^*(T)$ by $T^0_{\mathrm{rg}} = \Phi \circ \pi_r$. The condition $Y_T \subset Y$ implies that $T^0(f) = T^0_{\mathrm{rg}}(f)$ for $f \in C_0(E^0_{\mathrm{rg}} \setminus Y) \subset C_0(E^0_{\mathrm{rg}} \setminus Y_T)$. For $f \in C_0(E^0)$ and $g \in C_0(E^0_{\mathrm{rg}})$, we have

$$T^0(f)T^0_{\mathrm{rg}}(g) = T^0(f)\Phi(\pi_r(g)) = \Phi(\pi_r(f)\pi_r(g)) = T^0_{\mathrm{rg}}(fg).$$

We also have $T^0_{\mathrm{rg}}(g)T^0(f) = T^0_{\mathrm{rg}}(gf)$.

Lemma 3.6. *For $h \in C_0(E_Y^0)$, there exist $f \in C_0(E^0)$ and $g \in C_0(E^0_{\mathrm{rg}}) \subset C_0(E^0)$ such that $h(v) = f(v) + g(v)$ for $v \in E^0$ and $h(\omega(v)) = f(v)$ for $v \in Y$. The element $T^0(f) + T^0_{\mathrm{rg}}(g) \in C^*(T)$ does not depend on the choices of f and g satisfying the above two equations.*

Proof. Take $h \in C_0(E_Y^0)$. Define a function g_0 on Y by $g_0(v) = h(v) - h(\omega(v))$ for $v \in Y$. For a net $\{v_i\}$ in Y converges to an element in ∂Y, we have $\lim g_0(v_i) = \lim\big(h(v_i) - h(\omega(v_i))\big) = 0$. Hence we get $g_0 \in C_0(Y)$. Since Y is closed in E^0_{rg}, g_0 extends to a function $g \in C_0(E^0_{\mathrm{rg}})$ such that $g(v) = h(v) - h(\omega(v))$ for $v \in Y$. Define $f \in C_0(E^0)$ by $f(v) = h(v) - g(v)$ for $v \in E^0$. Now it is easy to see that f and g satisfy the equations $h(v) = f(v) + g(v)$ for $v \in E^0$ and $h(\omega(v)) = f(v)$ for $v \in Y$. Let us take other $f' \in C_0(E^0)$ and $g' \in C_0(E^0_{\mathrm{rg}})$ satisfying the two conditions. Then we have $g - g' = -(f - f') \in C_0(E^0_{\mathrm{rg}} \setminus Y)$. Hence we see

$$\begin{aligned}\big(T^0(f) + T^0_{\mathrm{rg}}(g)\big) - \big(T^0(f') + T^0_{\mathrm{rg}}(g')\big) &= T^0(f - f') + T^0_{\mathrm{rg}}(g - g')\\ &= T^0(f - f') - T^0_{\mathrm{rg}}(f - f') = 0.\end{aligned}$$

Thus the element $T^0(f) + T^0_{\mathrm{rg}}(g)$ does not depend on the choices of f and g satisfying the two conditions. □

For $h \in C_0(E_Y^0)$, we define $\widetilde{T}^0(h) \in C^*(T)$ by $\widetilde{T}^0(h) = T^0(f) + T^0_{\mathrm{rg}}(g)$ where $f \in C_0(E^0)$ and $g \in C_0(E^0_{\mathrm{rg}})$ are elements satisfying the two equations in Lemma 3.6.

Proposition 3.7. *The map $\widetilde{T}^0\colon C_0(E_Y^0) \to C^*(T)$ is a $*$-homomorphism satisfying the equation $T^0 = \widetilde{T}^0 \circ \mu_Y^0$.*

Proof. It is clear that $\widetilde{T}^0$ is linear and $*$-preserving. We will show that it is multiplicative. Take $h_1, h_2 \in C_0(E_Y^0)$. For $i = 1, 2$, choose $f_i \in C_0(E^0)$ and $g_i \in C_0(E^0_{\mathrm{rg}})$

satisfying the two conditions in Lemma 3.6 for $h_i \in C_0(E^0_Y)$. Set $f = f_1 f_2 \in C_0(E^0)$ and

$$g = (f_1 + g_1)(f_2 + g_2) - f_1 f_2 = f_1 g_2 + g_1 f_2 + g_1 g_2 \in C_0(E^0_{\mathrm{rg}}).$$

Then we have $(h_1 h_2)(v) = f(v) + g(v)$ for $v \in E^0$ and $(h_1 h_2)(\omega(v)) = f(v)$ for $v \in Y$. Hence we get $\widetilde{T}^0(h_1 h_2) = T^0(f) + T^0_{\mathrm{rg}}(g)$. On the other hand, we have

$$\begin{aligned}\widetilde{T}^0(h_1)\widetilde{T}^0(h_2) &= \big(T^0(f_1) + T^0_{\mathrm{rg}}(g_1)\big)\big(T^0(f_2) + T^0_{\mathrm{rg}}(g_2)\big) \\ &= T^0(f_1 f_2) + T^0_{\mathrm{rg}}(f_1 g_2 + g_1 f_2 + g_1 g_2) \\ &= T^0(f) + T^0_{\mathrm{rg}}(g)\end{aligned}$$

Thus $\widetilde{T}^0$ is multiplicative.

Take $f_0 \in C_0(E^0)$ and set $h = \mu^0_Y(f_0) \in C_0(E^0_Y)$. We can take $f = f_0$ and $g = 0$ in the definition of $\widetilde{T}^0(h)$. Hence we get $\widetilde{T}^0(h) = T^0(f_0)$. This proves $T^0 = \widetilde{T}^0 \circ \mu^0_Y$. □

Proposition 3.8. *The $*$-homomorphism $\widetilde{T}^0 \colon C_0(E^0_Y) \to C^*(T)$ is injective if and only if T is an injective Toeplitz E-pair such that $Y_T = Y$.*

Proof. Suppose that $\widetilde{T}^0$ is injective. Since $T^0 = \widetilde{T}^0 \circ \mu^0_Y$, the map T^0 is injective. For $g_0 \in C_0(E^0_{\mathrm{rg}})$ with $T^0(g_0) = T^0_{\mathrm{rg}}(g_0)$, define $h \in C_0(\omega(Y)) \subset C_0(E^0_Y)$ by $h(\omega(v)) = g_0(v)$. We can take $f = g_0$ and $g = -g_0$ in the definition of $\widetilde{T}^0(h)$. Hence we have $\widetilde{T}^0(h) = T^0(g_0) - T^0_{\mathrm{rg}}(g_0) = 0$, and so $h = 0$. This implies that $g_0 \in C_0(E^0_{\mathrm{rg}} \setminus Y)$. Thus we get $Y_T \supset Y$. Hence we have shown $Y_T = Y$ because the other inclusion is assumed.

Conversely take an injective Toeplitz E-pair T such that $Y_T = Y$. Take $h \in C_0(E^0_Y)$ with $\widetilde{T}^0(h) = 0$. Choose $f \in C_0(E^0)$ and $g \in C_0(E^0_{\mathrm{rg}})$ such that $h(v) = f(v) + g(v)$ for $v \in E^0$ and $h(\omega(v)) = f(v)$ for $v \in Y$. The condition $\widetilde{T}^0(h) = T^0(f) + T^0_{\mathrm{rg}}(g) = 0$ implies that $T^0(f) = T^0_{\mathrm{rg}}(-g) \in \varPhi\big(\mathcal{K}(C_d(E^1))\big)$. Hence by Lemma 3.2, we have $f \in C_0(E^0_{\mathrm{rg}} \setminus Y_T)$ and $T^0(f) = T^0_{\mathrm{rg}}(f)$. Since T^0 is injective, so is T^0_{rg}. Hence we have $f = -g$. Therefore we see $h(v) = f(v) + g(v) = 0$ for $v \in E^0$ and $h(\omega(v)) = f(v) = 0$ for $v \in Y = Y_T$. Thus we have $h = 0$. This shows that $\widetilde{T}^0$ is injective. We are done. □

Next, we define a linear map $\widetilde{T}^1 \colon C_{d_Y}(E^1_Y) \to C^*(T)$ such that $T^1 = \widetilde{T}^1 \circ \mu^1_Y$. Recall that the open subset E^1_{rg} of E^1 is defined by $E^1_{\mathrm{rg}} = d^{-1}(E^0_{\mathrm{rg}})$ and we showed that

$$C_d(E^1_{\mathrm{rg}}) = \{\xi g \in C_d(E^1) \mid \xi \in C_d(E^1), g \in C_0(E^0_{\mathrm{rg}})\}$$

in [K1, Lemma 1.12].

Lemma 3.9. *The map $T^1_{\mathrm{rg}} \colon C_d(E^1_{\mathrm{rg}}) \to C^*(T)$ defined by $T^1_{\mathrm{rg}}(\xi g) = T^1(\xi) T^0_{\mathrm{rg}}(g)$ is a well-defined linear map satisfying that*

$$T^1_{\mathrm{rg}}(\eta_1)^* T^1_{\mathrm{rg}}(\eta_2) = T^0_{\mathrm{rg}}(\langle \eta_1, \eta_2 \rangle)$$

for $\eta_1, \eta_2 \in C_d(E^1_{\mathrm{rg}})$.

Proof. For $i=1,2$, take $\eta_i=\xi_i g_i\in C_d(E^1_{\mathrm{rg}})$ with $\xi_i\in C_d(E^1)$ and $g_i\in C_0(E^0_{\mathrm{rg}})$. We see that

$$\begin{aligned}\big(T^1(\xi_1)T^0_{\mathrm{rg}}(g_1)\big)^*\big(T^1(\xi_2)T^0_{\mathrm{rg}}(g_2)\big)&=T^0_{\mathrm{rg}}(\overline{g_1})T^0(\langle\xi_1,\xi_2\rangle)T^0_{\mathrm{rg}}(g_2)\\&=T^0_{\mathrm{rg}}\big(\overline{g_1}\langle\xi_1,\xi_2\rangle g_2\big)\\&=T^0_{\mathrm{rg}}\big(\langle\eta_1,\eta_2\rangle\big).\end{aligned}$$

This proves the last equality. For $\eta\in C_d(E^1_{\mathrm{rg}})$, take $\xi_i\in C_d(E^1)$ and $g_i\in C_0(E^0_{\mathrm{rg}})$ with $\eta=\xi_i g_i$ for $i=1,2$. By the computation above, we have

$$\big(T^1(\xi_i)T^0_{\mathrm{rg}}(g_i)\big)^*\big(T^1(\xi_j)T^0_{\mathrm{rg}}(g_j)\big)=T^0_{\mathrm{rg}}\big(\langle\eta,\eta\rangle\big)$$

for $i,j=1,2$. Hence we get $x^*x=0$ for $x=T^1(\xi_1)T^0_{\mathrm{rg}}(g_1)-T^1(\xi_2)T^0_{\mathrm{rg}}(g_2)\in C^*(T)$. Thus we have $T^1(\xi_1)T^0_{\mathrm{rg}}(g_1)=T^1(\xi_2)T^0_{\mathrm{rg}}(g_2)$. This proves the well-definedness of T^1_{rg}. For $\eta_1,\eta_2\in C_d(E^1_{\mathrm{rg}})$, we have

$$\big(T^1_{\mathrm{rg}}(\eta_1+\eta_2)-T^1_{\mathrm{rg}}(\eta_1)-T^1_{\mathrm{rg}}(\eta_2)\big)^*\big(T^1_{\mathrm{rg}}(\eta_1+\eta_2)-T^1_{\mathrm{rg}}(\eta_1)-T^1_{\mathrm{rg}}(\eta_2)\big)=0$$

by the computation above. This proves the linearity of T^1_{rg}. □

Lemma 3.10. *For $\eta\in C_d(E^1_{\mathrm{rg}}\setminus d^{-1}(Y))\subset C_d(E^1_{\mathrm{rg}})$, we have $T^1(\eta)=T^1_{\mathrm{rg}}(\eta)$.*

Proof. For $\eta\in C_d(E^1_{\mathrm{rg}}\setminus d^{-1}(Y))$, we can find $\xi\in C_d(E^1)$ and $g\in C_0(E^0_{\mathrm{rg}}\setminus Y)$ such that $\eta=\xi g$ by [K1, Lemma 1.12]. Hence we have

$$T^1_{\mathrm{rg}}(\eta)=T^1(\xi)T^0_{\mathrm{rg}}(g)=T^1(\xi)T^0(g)=T^1(\xi g)=T^1(\eta).$$

We are done. □

By using Lemma 3.10, we can prove the following a proof of the following lemma may be given that is similar to the proof of Lemma 3.6.

Lemma 3.11. *For $\zeta\in C_{d_Y}(E^1_Y)$, there exist $\xi\in C_d(E^1)$ and $\eta\in C_d(E^1_{\mathrm{rg}})\subset C_d(E^1)$ such that $\zeta(e)=\xi(e)+\eta(e)$ for $e\in E^1$ and $\zeta(\omega(e))=\xi(e)$ for $e\in d^{-1}(Y)$. The element $T^1(\xi)+T^1_{\mathrm{rg}}(\eta)\in C^*(T)$ does not depend on the choices of ξ and η satisfying the above two equations.*

Hence we can define a linear map $\widetilde{T}^1\colon C_{d_Y}(E^1_Y)\to C^*(T)$ by $\widetilde{T}^1(\zeta)=T^1(\xi)+T^1_{\mathrm{rg}}(\eta)$ for $\zeta\in C_{d_Y}(E^1_Y)$, where $\xi\in C_d(E^1)$ and $\eta\in C_d(E^1_{\mathrm{rg}})$ satisfy the two conditions in Lemma 3.11. It is easy to see that $T^1=\widetilde{T}^1\circ\mu^1_Y$. We will prove that the pair $\widetilde{T}=(\widetilde{T}^0,\widetilde{T}^1)$ is a Cuntz-Krieger E_Y-pair.

Lemma 3.12. *For $f\in C_0(E^0)$, $g\in C_0(E^0_{\mathrm{rg}})$, $\xi\in C_d(E^1)$ and $\eta\in C_d(E^1_{\mathrm{rg}})$, we have*

$$\begin{gathered}T^0(f)T^1_{\mathrm{rg}}(\eta)=T^1_{\mathrm{rg}}(\pi_r(f)\eta),\quad T^0_{\mathrm{rg}}(g)T^1_{\mathrm{rg}}(\eta)=T^1_{\mathrm{rg}}(\pi_r(g)\eta),\\ \textit{and}\quad T^1(\xi)^*T^1_{\mathrm{rg}}(\eta)=T^0_{\mathrm{rg}}(\langle\xi,\eta\rangle).\end{gathered}$$

Proof. Straightforward. □

Lemma 3.13. *We have $\widetilde{T}^1(\zeta_1)^*\widetilde{T}^1(\zeta_2)=\widetilde{T}^0(\langle\zeta_1,\zeta_2\rangle)$ for $\zeta_1,\zeta_2\in C_{d_Y}(E^1_Y)$.*

Proof. For $i = 1, 2$, take $\xi_i \in C_d(E^1)$ and $\eta_i \in C_d(E^1_{\mathrm{rg}})$ satisfying that $\zeta_i(e) = \xi_i(e) + \eta_i(e)$ for $e \in E^1$ and $\zeta_i(\omega(e)) = \xi_i(e)$ for $e \in d^{-1}(Y)$. Set $f = \langle \xi_1, \xi_2 \rangle \in C_0(E^0)$ and

$$g = \langle \xi_1 + \eta_1, \xi_2 + \eta_2 \rangle - \langle \xi_1, \xi_2 \rangle = \langle \eta_1, \xi_2 \rangle + \langle \xi_1, \eta_2 \rangle + \langle \eta_1, \eta_2 \rangle \in C_0(E^0_{\mathrm{rg}}).$$

For $v \in E^0$, we have

$$\begin{aligned} \langle \zeta_1, \zeta_2 \rangle(v) &= \sum_{e \in (d_Y)^{-1}(v)} \overline{\zeta_1(e)} \zeta_2(e) = \sum_{e \in d^{-1}(v)} \overline{(\xi_1 + \eta_1)(e)} (\xi_2 + \eta_2)(e) \\ &= \langle \xi_1 + \eta_1, \xi_2 + \eta_2 \rangle(v) = f(v) + g(v). \end{aligned}$$

Similarly for $v \in Y$, we have

$$\begin{aligned} \langle \zeta_1, \zeta_2 \rangle(\omega(v)) &= \sum_{e \in (d_Y)^{-1}(\omega(v))} \overline{\zeta_1(e)} \zeta_2(e) = \sum_{e \in d^{-1}(v) \subset d^{-1}(Y)} \overline{\zeta_1(\omega(e))} \zeta_2(\omega(e)) \\ &= \sum_{e \in d^{-1}(v)} \overline{\xi_1(e)} \xi_2(e) = \langle \xi_1, \xi_2 \rangle(v) = f(v). \end{aligned}$$

Hence we have $\widetilde{T}^0(\langle \zeta_1, \zeta_2 \rangle) = T^0(f) + T^0_{\mathrm{rg}}(g)$. By Lemma 3.9 and Lemma 3.12, we have

$$\begin{aligned} \widetilde{T}^1(\zeta_1)^* \widetilde{T}^1(\zeta_2) &= \big(T^1(\xi_1) + T^1_{\mathrm{rg}}(\eta_1)\big)^* \big(T^1(\xi_2) + T^1_{\mathrm{rg}}(\eta_2)\big) \\ &= T^0(\langle \xi_1, \xi_2 \rangle) + T^0_{\mathrm{rg}}(\langle \eta_1, \xi_2 \rangle) + T^0_{\mathrm{rg}}(\langle \xi_1, \eta_2 \rangle) + T^0_{\mathrm{rg}}(\langle \eta_1, \eta_2 \rangle) \\ &= T^0(f) + T^0_{\mathrm{rg}}(g) \\ &= \widetilde{T}^0(\langle \zeta_1, \zeta_2 \rangle). \end{aligned}$$

□

Lemma 3.14. *We have $\widetilde{T}^0(h)\widetilde{T}^1(\zeta) = \widetilde{T}^1(\pi_{r_Y}(h)\zeta)$ for $h \in C_0(E^0_Y)$ and $\zeta \in C_{d_T}(E^1_Y)$.*

Proof. Take $f \in C_0(E^0)$ and $g \in C_0(E^0_{\mathrm{rg}})$ such that $h(v) = f(v) + g(v)$ for $v \in E^0$ and $h(\omega(v)) = f(v)$ for $v \in Y$. Take $\xi \in C_d(E^1)$ and $\eta \in C_d(E^1_{\mathrm{rg}})$ such that $\zeta(e) = \xi(e) + \eta(e)$ for $e \in E^1$ and $\zeta(\omega(e)) = \xi(e)$ for $e \in d^{-1}(Y)$. We have $T^0_{\mathrm{rg}}(g)T^1(\xi) = T^1(\pi_r(g)\xi)$. From this fact and Lemma 3.12, we see that

$$\begin{aligned} \widetilde{T}^0(h)\widetilde{T}^1(\zeta) &= \big(T^0(f) + T^0_{\mathrm{rg}}(g)\big)\big(T^1(\xi) + T^1_{\mathrm{rg}}(\eta)\big) \\ &= T^1(\pi_r(f+g)\xi) + T^1_{\mathrm{rg}}(\pi_r(f+g)\eta) \end{aligned}$$

Let us set $\xi' = \pi_r(f+g)\xi \in C_d(E^1)$ and $\eta' = \pi_r(f+g)\eta \in C_d(E^1_{\mathrm{rg}})$. For $e \in E^1$, we have

$$(\pi_{r_Y}(h)\zeta)(e) = h(r(e))\zeta(e) = \big(f(r(e)) + g(r(e))\big)\big(\xi(e) + \eta(e)\big) = \xi'(e) + \eta'(e).$$

For $e \in d^{-1}(Y)$, we have

$$(\pi_{r_Y}(h)\zeta)(\omega(e)) = h(r(e))\zeta(\omega(e)) = \big(f(r(e)) + g(r(e))\big)\xi(e) = \xi'(e).$$

Hence we get

$$\widetilde{T}^1(\pi_{r_Y}(h)\zeta) = T^1(\xi') + T^1_{\mathrm{rg}}(\eta') = \widetilde{T}^0(h)\widetilde{T}^1(\zeta).$$

The proof is completed. □

Proposition 3.15. *The pair $\widetilde{T} = (\widetilde{T}^0, \widetilde{T}^1)$ is a Cuntz-Krieger E_Y-pair satisfying the equation $T^i = \widetilde{T}^i \circ \mu_Y^i$ for $i = 0, 1$ and $C^*(\widetilde{T}) = C^*(T)$.*

Proof. By Lemma 3.13 and Lemma 3.14, $\widetilde{T}$ is a Toeplitz E_Y-pair. We will show that $\widetilde{T}$ is a Cuntz-Krieger E_Y-pair. To do so, it suffices to see that $\widetilde{T}^0\big(C_0((E_Y^0)_{\mathrm{rg}})\big) \subset \widetilde{\Phi}\big(\mathcal{K}(C_{d_Y}(E_Y^1))\big)$ by Lemma 3.2. Take $h \in C_0((E_Y^0)_{\mathrm{rg}})$. By Lemma 3.3, we have $(E_Y^0)_{\mathrm{rg}} = E_{\mathrm{rg}}^0$. Hence there exists $g \in C_0(E_{\mathrm{rg}}^0)$ such that $h(v) = g(v)$ for $v \in E^0$. Thus we have $\widetilde{T}^0(h) = T^0(0) + T_{\mathrm{rg}}^0(g) = \Phi(\pi_r(g))$. Since $T^1(C_d(E^1)) \subset \widetilde{T}^1(C_{d_Y}(E_Y^1))$, we have $\Phi\big(\mathcal{K}(C_d(E^1))\big) \subset \widetilde{\Phi}\big(\mathcal{K}(C_{d_Y}(E_Y^1))\big)$. Hence we get $\widetilde{T}^0(h) \in \widetilde{\Phi}\big(\mathcal{K}(C_{d_Y}(E_Y^1))\big)$ for every $h \in C_0((E_Y^0)_{\mathrm{rg}})$. Thus $\widetilde{T}$ is a Cuntz-Krieger E_Y-pair. As we have already seen, the two equations $T^i = \widetilde{T}^i \circ \mu_Y^i$ hold for $i = 0, 1$. This implies $\widetilde{T}^0(C_0(E_Y^0)) \supset T^0(C_0(E^0))$ and $\widetilde{T}^1(C_{d_Y}(E_Y^1)) \supset T^1(C_d(E^1))$. Hence we have $C^*(\widetilde{T}) = C^*(T)$. □

By Proposition 3.15, we have a surjective $*$-homomorphism $\rho_T \colon \mathcal{O}(E_Y) \to C^*(T)$ such that $\widetilde{T}^i = \rho_T \circ t_Y^i$ for $i = 0, 1$. We have

$$\rho_T \circ T_Y^i = \rho_T \circ t_Y^i \circ \mu^i = \widetilde{T}^i \circ \mu^i = T^i,$$

for $i = 0, 1$. We study for which Toeplitz pairs T the surjections ρ_T are injective.

A *loop* is an element $e \in E^n$ with $n \geq 1$ such that $d(e) = r(e)$, and the vertex $d(e) = r(e)$ is called the *base point* of the loop e. A loop $e = (e_1, \ldots, e_n)$ is said to be *without entrances* if $r^{-1}(r(e_k)) = \{e_k\}$ for $k = 1, \ldots, n$. Recall that a topological graph E is said to be *topologically free* if the set of base points of loops without entrances has an empty interior ([K1, Definition 5.4]).

Proposition 3.16. *When E is topologically free, the $*$-homomorphism ρ_T is an isomorphism if and only if the Toeplitz E-pair T is injective and satisfies $Y_T = Y$.*

Proof. When E is topologically free, the topological graph E_Y is also topologically free because vertices in $\omega(Y)$ receive no edges. Hence ρ_T is an isomorphism if and only if $\widetilde{T}$ is injective by [K1, Theorem 5.12]. By Proposition 3.8, $\widetilde{T}$ is injective whenever T is an injective Toeplitz E-pair such that $Y_T = Y$. This completes the proof. □

Lemma 3.17. *The Cuntz-Krieger E_Y-pair $\widetilde{T}$ admits a gauge action if and only if T does.*

Proof. Since $\widetilde{T}^0(C_0(E_Y^0)) \supset T^0(C_0(E^0))$ and $\widetilde{T}^1(C_{d_Y}(E_Y^1)) \supset T^1(C_d(E^1))$, it is clear that if $\widetilde{T}$ admits a gauge action then T also does. Conversely, suppose that for each $z \in \mathbb{T}$, there exists an automorphism β_z on $C^*(T)$ such that $\beta_z(T^0(f)) = T^0(f)$ and $\beta_z(T^1(\xi)) = zT^1(\xi)$ for $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$. Then it is easy to see that $\beta_z(T_{\mathrm{rg}}^0(g)) = T_{\mathrm{rg}}^0(g)$ and $\beta_z(T_{\mathrm{rg}}^1(\eta)) = zT_{\mathrm{rg}}^1(\eta)$ for $g \in C_0(E_{\mathrm{rg}}^0)$ and $\eta \in C_d(E_{\mathrm{rg}}^1)$. Hence $\beta_z(\widetilde{T}^0(h)) = \widetilde{T}^0(h)$ and $\beta_z(\widetilde{T}^1(\zeta)) = z\widetilde{T}^1(\zeta)$ for $h \in C_0(E_Y^0)$ and $\zeta \in C_{d_Y}(E_Y^1)$. Thus the pair $\widetilde{T}$ admits a gauge action. We are done. □

Proposition 3.18. *The $*$-homomorphism ρ_T is an isomorphism if and only if the Toeplitz E-pair T is injective, admits a gauge action, and satisfies $Y_T = Y$.*

Proof. This follows from Proposition 1.6 with the help of Lemma 3.17 in a similar way to the proof of Proposition 3.16. □

Corollary 3.19. *Let E be a topological graph. For an injective Toeplitz E-pair T which admits a gauge action, the C^*-algebra $C^*(T)$ is isomorphic to $\mathcal{O}(E_{Y_T})$.*

Proposition 3.20. *The Toeplitz E-pair T_Y on $\mathcal{O}(E_Y)$ in Proposition 3.5 satisfies $C^*(T_Y) = \mathcal{O}(E_Y)$.*

Proof. Since we have $Y_{T_Y} = Y$ by Proposition 3.5, we get a $*$-homomorphism $\rho_{T_Y}\colon \mathcal{O}(E_Y) \to C^*(T_Y)$ satisfying the equation $\rho_T \circ T_Y^i = T_Y^i$ for $i = 0, 1$. We have $\rho_{T_Y}(x) = x$ for $x \in C^*(T_Y) \subset \mathcal{O}(E_Y)$. By Proposition 3.18, ρ_{T_Y} is an isomorphism. Thus we have $C^*(T_Y) = \mathcal{O}(E_Y)$. $\square$

Now we have the following proposition which implies that the C^*-algebra $\mathcal{O}(E_Y)$ is the universal C^*-algebra generated by Toeplitz E-pairs T satisfying $Y_T \subset Y$.

Proposition 3.21. *Let E be a topological graph, and Y be a closed subset of E^0_{rg}. For a Toeplitz E-pair T with $Y_T \subset Y$, there exists a unique surjective $*$-homomorphism $\rho_T\colon \mathcal{O}(E_Y) \to C^*(T)$ satisfying $T^i = \rho_T \circ T_Y^i$ for $i = 0, 1$.*

Proof. We have already seen that there exists such a surjection ρ_T. The uniqueness follows from Proposition 3.20. $\square$

Corollary 3.22. *For a topological graph E, we have $\mathcal{T}(E) \cong \mathcal{O}(E_{E^0_{\mathrm{rg}}})$.*

Proof. Take $Y = E^0_{\mathrm{rg}}$ in Proposition 3.21. $\square$

As a consequence of the analysis above, we get the following proposition which means that $\mathcal{O}(E)$ is the smallest C^*-algebra among C^*-algebras generated by injective Toeplitz E-pairs which admit gauge actions. Thus we get an alternative definition of $\mathcal{O}(E)$ which does not use the space E^0_{rg} or Cuntz-Krieger pairs.

Proposition 3.23. *Let E be a topological graph, and T be an injective Toeplitz E-pair which admits a gauge action. Then there exists a surjective $*$-homomorphism $\Psi\colon C^*(T) \to \mathcal{O}(E)$ such that $\Psi \circ T^i = t^i$ for $i = 0, 1$.*

Proof. We have the isomorphism $C^*(T) \cong \mathcal{O}(E_{Y_T})$ by Corollary 3.19, and we have the surjection $\mathcal{O}(E_{Y_T}) \to \mathcal{O}(E)$ by Proposition 3.21. It is routine to check that the composition $\Psi\colon C^*(T) \to \mathcal{O}(E)$ of the above two maps satisfies $\Psi \circ T^i = t^i$ for $i = 0, 1$. $\square$

Remark 3.24. The surjection $\mathcal{O}(E_{Y_T}) \to \mathcal{O}(E)$ in the proof of Proposition 3.23 is induced by the regular factor map $m = (m^0, m^1)$ from E to E_{Y_T} defined by the embeddings $m^0\colon E^0 \to F^0_T$ and $m^1\colon E^1 \to F^1_T$.

We finish this section by generalizing Corollary 3.19 to all Toeplitz pairs admitting gauge actions. Let us fix a topological graph $E = (E^0, E^1, d, r)$ and a Toeplitz E-pair T. We define a closed subset X^0 of E^0 by $\ker T^0 = C_0(E^0 \setminus X^0)$. We set $X^1 = d^{-1}(X^0)$ which is a closed subset of E^1.

Proposition 3.25. *We have $r(X^1) \subset X^0$.*

Proof. Take $e \in X^1$. We can find $\xi \in C_d(E^1)$ with $\xi(e) = 1$ and $\xi(e') = 0$ for all $e' \in d^{-1}(d(e)) \setminus \{e\}$ because e is isolated in $d^{-1}(d(e))$. For any $f \in \ker T^0$, we have

$$T^0(\langle \xi, \pi_r(f)\xi\rangle) = T^1(\xi)^* T^0(f) T^1(\xi) = 0$$

Hence $\langle \xi, \pi_r(f)\xi\rangle \in \ker T^0$. We get $\langle \xi, \pi_r(f)\xi\rangle(d(e)) = 0$ for all $f \in \ker T^0$. Since

$$\langle \xi, \pi_r(f)\xi\rangle(d(e)) = \sum_{e' \in d^{-1}(d(e))} \overline{\xi(e')} f(r(e'))\xi(e') = f(r(e)),$$

we have $f(r(e)) = 0$ for all $f \in \ker T^0$. This implies $r(e) \in X^0$. Thus $r(X^1) \subset X^0$. □

By Proposition 3.25, the quadruple $X = (X^0, X^1, d, r)$ is a topological graph. For $\xi \in C_d(E^1 \setminus X^1)$, we can find $\eta \in C_d(E^1)$ and $f \in C_0(E^0 \setminus X^0)$ with $\xi = \eta f$. Hence we have $T^1(\xi) = T^1(\eta)T^0(f) = 0$. Thus the maps $T^0 \colon C_0(E^0) \to C^*(T)$ and $T^1 \colon C_d(E^1) \to C^*(T)$ factor through $\dot{T}^0 \colon C_0(X^0) \to C^*(T)$ and $\dot{T}^1 \colon C_d(X^1) \to C^*(T)$. It is easy to see the following.

Lemma 3.26. *The pair $\dot{T} = (\dot{T}^0, \dot{T}^1)$ is an injective Toeplitz X-pair with $C^*(\dot{T}) = C^*(T)$. The pair $\dot{T}$ admits a gauge action if and only if T does.*

We define a closed subset Y of X^0_{rg} by

$$C_0(X^0_{\mathrm{rg}} \setminus Y) = \{f \in C_0(X^0_{\mathrm{rg}}) \mid \dot{T}^0(f) = \dot{\Phi}(\pi_r(f))\}.$$

where the $*$-homomorphism $\dot{\Phi} \colon \mathcal{K}(C_d(X^1)) \to C^*(T)$ is defined from the pair $\dot{T}$. The following proposition easily follows from Corollary 3.19.

Proposition 3.27. *Let E be a topological graph. For a Toeplitz E-pair T which admits a gauge action, the C^*-algebra $C^*(T)$ is isomorphic to $\mathcal{O}(X_Y)$ where the topological graph X_Y is obtained from the topological graph X by attaching extra vertices isomorphic to Y and extra edges isomorphic to $d^{-1}(Y)$ as above.*

Remark 3.28. For a Toeplitz E-pair T which does not admit a gauge action, we just get an injective Cuntz-Krieger X_Y-pair T' on $C^*(T)$ such that $C^*(T') = C^*(T)$.

Remark 3.29. In general, the Toeplitz X-pair $\dot{T}$ defined above may not be a Cuntz-Krieger pair even when T is a Cuntz-Krieger E-pair. This phenomena will be studied in the analysis of the ideal structures of $\mathcal{O}(E)$ in [K5].

4. Projective systems of topological graphs

In this section, we define projective systems of topological graphs and their projective limits, and study how these relate to C^*-algebras $\mathcal{T}(E)$ and $\mathcal{O}(E)$.

Definition 4.1. A *projective system of topological graphs* over a directed set Λ consists of a set of topological graphs $E_\lambda = (E^0_\lambda, E^1_\lambda, d_\lambda, r_\lambda)$ for $\lambda \in \Lambda$ and a set of factor maps $m_{\lambda,\lambda'} \colon E_{\lambda'} \to E_\lambda$ for $\lambda, \lambda' \in \Lambda$ with $\lambda \preceq \lambda'$ satisfying the equations $m_{\lambda,\lambda} = \mathrm{id}_{E_\lambda}$ for $\lambda \in \Lambda$ and $m_{\lambda,\lambda'} \circ m_{\lambda',\lambda''} = m_{\lambda,\lambda''}$ for $\lambda \preceq \lambda' \preceq \lambda''$.

Let us take a projective system of topological graphs $(\{E_\lambda\}_{\lambda\in\Lambda}, \{m_{\lambda,\lambda'}\}_{\lambda\preceq\lambda'})$ and fix it. For $i = 0, 1$, we define a compact set $\widetilde{E}^i$ by the projective limit of the compact sets $\widetilde{E}^i_\lambda$ by the maps $m^i_{\lambda,\lambda'}$. For each $\lambda \in \Lambda$ and $i = 0, 1$, we denote by m^i_λ the natural continuous map from $\widetilde{E}^i$ to $\widetilde{E}^i_\lambda$. For $i = 0, 1$, we denote by ∞ the element $x \in \widetilde{E}^i$ with $m^i_\lambda(x) = \infty$ for all $\lambda \in \Lambda$, and set $E^i = \widetilde{E}^i \setminus \{\infty\}$. The set E^i is a locally compact space whose one-point compactification is $\widetilde{E}^i$. For any $v \in E^0$, there exists $\lambda_0 \in \Lambda$ such that $m^0_{\lambda_0}(v) \in E^0_{\lambda_0}$. Then we have $m^0_\lambda(v) \in E^0_\lambda$ for

any $\lambda \succeq \lambda_0$. The net $\{m^0_\lambda(v)\}_{\lambda\succeq\lambda_0}$ satisfies the equation $m^0_{\lambda,\lambda'}(m^0_{\lambda'}(v)) = m^0_\lambda(v)$ for $\lambda' \succeq \lambda \succeq \lambda_0$. Conversely, for $\lambda_0 \in \Lambda$, a net $\{v_\lambda\}_{\lambda\succeq\lambda_0}$ with $v_\lambda \in E^0_\lambda$ satisfying the equation $m^0_{\lambda,\lambda'}(v_{\lambda'}) = v_\lambda$ for $\lambda' \succeq \lambda \succeq \lambda_0$, gives an element in E^0. Thus elements in E^0 are represented by such nets. Elements in E^1 are represented similarly. We define maps $d, r\colon E^1 \to E^0$ by $d(e) = \{d_\lambda(e_\lambda)\}_{\lambda\succeq\lambda_0}$ and $r(e) = \{r_\lambda(e_\lambda)\}_{\lambda\succeq\lambda_0}$ for $e = \{e_\lambda\}_{\lambda\succeq\lambda_0} \in E^1$. This is well-defined because $m_{\lambda,\lambda'}$'s are factor maps. To prove that the quadruple $E = (E^0, E^1, d, r)$ is a topological graph, we need the following lemma.

Lemma 4.2. *Let $m = (m^0, m^1)$ be a factor map from a topological graph $E = (E^0, E^1, d_E, r_E)$ to a topological graph $F = (F^0, F^1, d_F, r_F)$. If the restriction of d_E to an open set $U \subset E^1$ is a homeomorphism onto the open set $d_E(U) \subset E^0$, then the restriction of d_F to $(m^1)^{-1}(U) \subset F^1$ is a homeomorphism onto $(m^0)^{-1}(d_E(U)) \subset F^0$.*

Proof. Take such an open subset $U \subset E^1$ and set $V = d_E(U)$. Since a bijective local homeomorphism is a homeomorphism, it suffices to see that the restriction of d_F to $(m^1)^{-1}(U) \subset F^1$ is a bijection onto $(m^0)^{-1}(V) \subset F^0$. For $e \in (m^1)^{-1}(U)$, we have $m^0(d_F(e)) = d_E(m^1(e)) \in V$ by the condition (i) in Definition 2.1. Hence $d_F((m^1)^{-1}(U)) \subset (m^0)^{-1}(V)$. Take $v \in (m^0)^{-1}(V)$. Since $m^0(v) \in V$, there exists unique $e' \in U$ such that $d_E(e') = m^0(v)$. By the condition (ii) in Definition 2.1, there exists unique $e \in F^1$ such that $m^1(e) = e'$ and $d_F(e) = v$. This $e \in F^1$ is the unique element in $(m^1)^{-1}(U)$ with $d_F(e) = v$. Hence the restriction of d_F to $(m^1)^{-1}(U)$ is a bijection onto $(m^0)^{-1}(V)$. We are done. $\square$

Proposition 4.3. *The quadruple $E = (E^0, E^1, d, r)$ defined from a projective system of topological graphs $(\{E_\lambda\}_{\lambda\in\Lambda}, \{m_{\lambda,\lambda'}\}_{\lambda\preceq\lambda'})$ as above is a topological graph.*

Proof. For $\lambda_0 \in \Lambda$ and an open subset V of $E^0_{\lambda_0}$, the set

$$r^{-1}\big((m^0_{\lambda_0})^{-1}(V)\big) = \bigcup_{\lambda\succeq\lambda_0} (m^1_\lambda)^{-1}\big(r_\lambda^{-1}(V)\big)$$

is an open subset of E^1. Since the family

$$\{(m^0_{\lambda_0})^{-1}(V) \mid \lambda_0 \in \Lambda,\ V \text{ is an open subset of } E^0_{\lambda_0}\}$$

is a basis of E^0, we see that $r\colon E^1 \to E^0$ is continuous. We will show that d is locally homeomorphic. Take $e = \{e_\lambda\}_{\lambda\succeq\lambda_0} \in E^1$. Take a neighborhood U_{λ_0} of $e_{\lambda_0} \in E^1_{\lambda_0}$ such that the restriction of d_{λ_0} to U_{λ_0} is a homeomorphism onto $d_{\lambda_0}(U_{\lambda_0})$. Set $V_{\lambda_0} = d_{\lambda_0}(U_{\lambda_0})$ which is a neighborhood of $d_{\lambda_0}(e_{\lambda_0}) \in E^0_{\lambda_0}$. For $\lambda \succeq \lambda_0$, we set $U_\lambda = (m^1_{\lambda_0,\lambda})^{-1}(U_{\lambda_0})$ and $V_\lambda = (m^0_{\lambda_0,\lambda})^{-1}(V_{\lambda_0})$. By Lemma 4.2, the restriction of d_λ to U_λ is a homeomorphism onto V_λ. Set $U = (m^1_{\lambda_0})^{-1}(U_{\lambda_0})$ and $V = (m^0_{\lambda_0})^{-1}(V_{\lambda_0})$. We see that U is a neighborhood of $e \in E^1$ and V is a neighborhood of $d(e) \in E^0$ because $m^0_{\lambda_0}(d(e)) = d_{\lambda_0}(e_{\lambda_0})$. We will show that the restriction of d to U is a homeomorphism onto V. Take $\lambda \succeq \lambda_0$ and open subsets $V' \subset V_\lambda$ and $U' \subset U_\lambda$ with $V' = d_\lambda(U')$. For $e' \in (m^1_\lambda)^{-1}(U') \subset U$, we have

$$m^0_\lambda(d(e')) = d_\lambda(m^1_\lambda(e')) \in V'.$$

Hence $d((m^1_\lambda)^{-1}(U')) \subset (m^0_\lambda)^{-1}(V')$. Take $v' \in (m^0_\lambda)^{-1}(V') \subset V$. Since the restriction of d_λ to U_λ is a bijection onto V_λ and $m^0_\lambda(v') \in V' \subset V_\lambda$, there exists unique $e'_\lambda \in U_\lambda$ with $d_\lambda(e'_\lambda) = m^0_\lambda(v')$. We have $e'_\lambda \in U'$. For each $\lambda' \succeq \lambda$ we

have $m^0_{\lambda'}(v') \in V_{\lambda'}$. Hence there exists unique $e'_{\lambda'} \in U_{\lambda'}$ with $d_{\lambda'}(e'_{\lambda'}) = m^0_{\lambda'}(v')$. By the uniqueness of $e'_{\lambda'}$, we have $m^1_{\lambda',\lambda''}(e'_{\lambda''}) = e'_{\lambda'}$ for $\lambda'' \succeq \lambda' \succeq \lambda$. Thus $e' = \{e'_{\lambda'}\}_{\lambda'\succeq\lambda} \in (m^1_\lambda)^{-1}(U') \subset U$ is the unique element in U with $d(e') = v'$. Hence the restriction of d to $(m^1_\lambda)^{-1}(U')$ is a bijection onto $(m^0_\lambda)^{-1}(V')$. Since the family

$$\{(m^1_\lambda)^{-1}(U') \mid \lambda \succeq \lambda_0,\ U' \text{ is an open subset of } U_\lambda\}$$

is a basis of U, and the family

$$\{(m^0_\lambda)^{-1}(V') \mid \lambda \succeq \lambda_0,\ V' \text{ is an open subset of } V_\lambda\}$$

is a basis of V, we see that the restriction of d to U is a homeomorphism onto V. Thus $d\colon E^1 \to E^0$ is a local homeomorphism. □

Definition 4.4. The topological graph E in Proposition 4.3 is called the *projective limit* of the projective system $(\{E_\lambda\}_{\lambda\in\Lambda}, \{m_{\lambda,\lambda'}\}_{\lambda\preceq\lambda'})$, and denoted by

$$\varprojlim \big(\{E_\lambda\}_{\lambda\in\Lambda}, \{m_{\lambda,\lambda'}\}_{\lambda\preceq\lambda'}\big),$$

or simply by $\varprojlim E_\lambda$.

Proposition 4.5. *For each $\lambda_0 \in \Lambda$, the pair $m_{\lambda_0} = (m^0_{\lambda_0}, m^1_{\lambda_0})$ is a factor map from $E = \varprojlim E_\lambda$ to E_{λ_0}.*

Proof. By the definitions of d and r, if $e \in E^1$ satisfies $m^1_{\lambda_0}(e) \in E^1_{\lambda_0}$, then we have $d_{\lambda_0}(m^1_{\lambda_0}(e)) = m^0_{\lambda_0}(d(e))$ and $r_{\lambda_0}(m^1_{\lambda_0}(e)) = m^0_{\lambda_0}(r(e))$. Take $e_{\lambda_0} \in E^1_{\lambda_0}$ and $v \in E^0$ such that $d_{\lambda_0}(e_{\lambda_0}) = m^0_{\lambda_0}(v)$. For each $\lambda \succeq \lambda_0$, we have $d_{\lambda_0}(e_{\lambda_0}) = m^0_{\lambda_0,\lambda}(m^0_\lambda(v))$. Since $m_{\lambda_0,\lambda}$ is a factor map, there exists unique $e_\lambda \in E^1_\lambda$ with $m^1_{\lambda_0,\lambda}(e_\lambda) = e_{\lambda_0}$ and $d_\lambda(e_\lambda) = m^0_\lambda(v)$. The uniqueness implies that $m^1_{\lambda,\lambda'}(e_{\lambda'}) = e_\lambda$ for $\lambda' \succeq \lambda \succeq \lambda_0$. Hence $e = \{e_\lambda\}_{\lambda\succeq\lambda_0}$ is the unique element in E^1 satisfying the equations $m^1_{\lambda_0}(e) = e_{\lambda_0}$ and $d(e) = v$. We are done. □

Denote by $T_\lambda = (T^0_\lambda, T^1_\lambda)$ the universal Toeplitz E_λ-pair in $\mathcal{T}(E_\lambda)$. For $\lambda \preceq \lambda'$, the factor map $m_{\lambda,\lambda'}$ gives a $*$-homomorphism $\mu^0_{\lambda,\lambda'}\colon C_0(E^0_\lambda) \to C_0(E^0_{\lambda'})$, a linear map $\mu^1_{\lambda,\lambda'}\colon C_{d_\lambda}(E^1_\lambda) \to C_{d_{\lambda'}}(E^1_{\lambda'})$ and a $*$-homomorphism $\mu_{\lambda,\lambda'}\colon \mathcal{T}(E_\lambda) \to \mathcal{T}(E_{\lambda'})$ such that $\mu_{\lambda,\lambda'} \circ T^i_\lambda = T^i_{\lambda'} \circ \mu^i_{\lambda,\lambda'}$ for $i = 0, 1$ by Proposition 2.3. For $\lambda \preceq \lambda' \preceq \lambda''$, we have $\mu^i_{\lambda',\lambda''} \circ \mu^i_{\lambda,\lambda'} = \mu^i_{\lambda,\lambda''}$ for $i = 0, 1$ and $\mu_{\lambda',\lambda''} \circ \mu_{\lambda,\lambda'} = \mu_{\lambda,\lambda''}$ by Proposition 2.4. For each $\lambda \in \Lambda$, the factor map m_λ gives a $*$-homomorphism $\mu^0_\lambda\colon C_0(E^0_\lambda) \to C_0(E^0)$, a linear map $\mu^1_\lambda\colon C_{d_\lambda}(E^1_\lambda) \to C_d(E^1)$ and a $*$-homomorphism $\mu_\lambda\colon \mathcal{T}(E_\lambda) \to \mathcal{T}(E)$ by Proposition 4.5. Since we have $\mu_{\lambda'} \circ \mu_{\lambda,\lambda'} = \mu_\lambda$ for $\lambda \preceq \lambda'$, $\{\mu_\lambda\}_{\lambda\in\Lambda}$ gives us a $*$-homomorphism $\varinjlim \mathcal{T}(E_\lambda) \to \mathcal{T}(E)$. We can naturally consider $C_0(E^0) = \varinjlim C_0(E^0_\lambda)$ and $C_d(E^1) = \varinjlim C_{d_\lambda}(E^1_\lambda)$. There exist a $*$-homomorphism $T^0\colon C_0(E^0) \to \varinjlim \mathcal{T}(E_\lambda)$ and a linear map $T^1\colon C_d(E^1) \to \varinjlim \mathcal{T}(E_\lambda)$ such that $T^i \circ \mu^i_\lambda = \mu_\lambda \circ T^i_\lambda$ for $i = 0, 1$ and $\lambda \in \Lambda$. One can easily see that the pair $T = (T^0, T^1)$ is a Toeplitz E-pair. By the universality of $\mathcal{T}(E)$, there exists a $*$-homomorphism $\mathcal{T}(E) \to \varinjlim \mathcal{T}(E_\lambda)$. It is easy to verify that the two maps $\varinjlim \mathcal{T}(E_\lambda) \to \mathcal{T}(E)$ and $\mathcal{T}(E) \to \varinjlim \mathcal{T}(E_\lambda)$ are the inverses of each others. Thus we get the following.

Proposition 4.6. *For a projective system $(\{E_\lambda\}_{\lambda\in\Lambda}, \{m_{\lambda,\lambda'}\}_{\lambda\preceq\lambda'})$, we have*

$$\mathcal{T}\big(\varprojlim E_\lambda\big) \cong \varinjlim \mathcal{T}(E_\lambda).$$

We define the reguraliry of projective limits and seek an analogous result of Proposition 4.6 for $\mathcal{O}(E)$.

Definition 4.7. A projective system $(\{E_\lambda\}_{\lambda\in\Lambda}, \{m_{\lambda,\lambda'}\}_{\lambda\preceq\lambda'})$ of topological graphs is said to be *regular* if $m_{\lambda,\lambda'}$ is regular for all $\lambda, \lambda' \in \Lambda$ with $\lambda \preceq \lambda'$.

Take a regular projective system $(\{E_\lambda\}_{\lambda\in\Lambda}, \{m_{\lambda,\lambda'}\}_{\lambda\preceq\lambda'})$ of topological graphs, and denote by $E = (E^0, E^1, d, r)$ its projective limit $\varprojlim E_\lambda$. Let $m_{\lambda_0} = (m^0_{\lambda_0}, m^1_{\lambda_0})$ be the natural factor map from $E = \varprojlim E_\lambda$ to E_{λ_0} for each $\lambda_0 \in \Lambda$.

Proposition 4.8. *For each $\lambda_0 \in \Lambda$, the factor map m_{λ_0} is regular.*

Proof. Take $v \in E^0$ with $m^0_{\lambda_0}(v) \in (E^0_{\lambda_0})_{\mathrm{rg}}$, and we will show that $r^{-1}(v)$ is a non-empty subset of $(m^1_{\lambda_0})^{-1}(E^1_{\lambda_0})$. Let e be an element in $r^{-1}(v)$. Take $\lambda \in \Lambda$ with $m^1_\lambda(e) \in E^1_\lambda$. We may assume $\lambda \succeq \lambda_0$. Since the factor map $m_{\lambda_0,\lambda}$ is regular and

$$m^0_{\lambda_0,\lambda}(r_\lambda(m^1_\lambda(e))) = m^0_{\lambda_0,\lambda}(m^0_\lambda(r(e))) = m^0_{\lambda_0}(v) \in (E^0_{\lambda_0})_{\mathrm{rg}},$$

we have $m^1_\lambda(e) \in (m^1_{\lambda_0,\lambda})^{-1}(E^1_{\lambda_0})$. Thus we have $e \in (m^1_{\lambda_0})^{-1}(E^1_{\lambda_0})$. Hence $r^{-1}(v) \subset (m^1_{\lambda_0})^{-1}(E^1_{\lambda_0})$. We will show that $r^{-1}(v) \neq \emptyset$.

Set $\Lambda_0 = \{\lambda \in \Lambda \mid \lambda \succeq \lambda_0\}$. For each $\lambda \in \Lambda_0$, we define $G_\lambda \subset E^1_\lambda$ by $G_\lambda = r_\lambda^{-1}(m^0_\lambda(v))$. Lemma 2.7 implies $m^0_\lambda(v) \in (E^0_\lambda)_{\mathrm{rg}}$ for $\lambda \in \Lambda_0$ because $m^0_{\lambda_0,\lambda}(m^0_\lambda(v)) = m^0_{\lambda_0}(v) \in (E^0_{\lambda_0})_{\mathrm{rg}}$. Hence G_λ is a non-empty compact set. We define a compact set G by $G = \prod_{\lambda\in\Lambda_0} G_\lambda$. For $\lambda, \lambda' \in \Lambda_0$ with $\lambda \preceq \lambda'$, we define a subset

$$F_{\lambda,\lambda'} = \big\{\{e_\lambda\}_{\lambda\in\Lambda_0} \in G \;\big|\; m_{\lambda,\lambda'}(e_{\lambda'}) = e_\lambda\big\}$$

It is clear that $F_{\lambda,\lambda'}$ is a closed subset of the compact set G for $\lambda, \lambda' \in \Lambda_0$ with $\lambda \preceq \lambda'$. We will show that $\bigcap_{k=1}^n F_{\lambda_k,\lambda'_k} \neq \emptyset$ for $\lambda_k, \lambda'_k \in \Lambda_0$ with $\lambda_k \preceq \lambda'_k$ for $k = 1, 2, \ldots, n$. We can find $\lambda'_0 \in \Lambda_0$ with $\lambda'_k \preceq \lambda'_0$ for $k = 1, 2, \ldots, n$. Since $G_{\lambda'_0} \neq \emptyset$, we can take $e_{\lambda'_0} \in G_{\lambda'_0}$. By the reguraliry of m_{λ,λ'_0}, we have $m_{\lambda,\lambda'_0}(e_{\lambda'_0}) \in G_\lambda$ for $\lambda \in \Lambda_0$ with $\lambda \preceq \lambda'_0$. Thus we can find $\{e_\lambda\}_{\lambda\in\Lambda_0} \in G$ such that $e_\lambda = m_{\lambda,\lambda'_0}(e_{\lambda'_0})$ with $\lambda \in \Lambda_0$ with $\lambda \preceq \lambda'_0$. For $k = 1, 2, \ldots, n$, we have

$$m_{\lambda_k,\lambda'_k}(e_{\lambda'_k}) = m_{\lambda_k,\lambda'_k}(m_{\lambda'_k,\lambda'_0}(e_{\lambda'_0})) = m_{\lambda_k,\lambda'_0}(e_{\lambda'_0}) = e_{\lambda_k}.$$

Hence we have shown that

$$\{e_\lambda\}_{\lambda\in\Lambda_0} \in \bigcap_{k=1}^n F_{\lambda_k,\lambda'_k} \neq \emptyset$$

for $\lambda_k, \lambda'_k \in \Lambda_0$ with $\lambda_k \preceq \lambda'_k$ for $k = 1, 2, \ldots, n$. Since G is compact, we can find

$$\{e_\lambda\}_{\lambda\in\Lambda_0} \in \bigcap_{\lambda\preceq\lambda'} F_{\lambda,\lambda'}.$$

Since we have $m_{\lambda,\lambda'}(e_{\lambda'}) = e_\lambda$ for $\lambda, \lambda' \in \Lambda_0$ with $\lambda \preceq \lambda'$. we get an element $e = \{e_\lambda\}_{\lambda\in\Lambda_0} \in E^1$ which satisfies $r(e) = v$. Hence $r^{-1}(v) \neq \emptyset$. The proof is completed. □

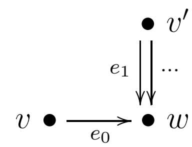
As in the case of $\mathcal{T}(E)$, we get the following commutative diagram $(\lambda \preceq \lambda')$:

$$
\begin{tikzcd}
C_0(E^0_\lambda) \arrow[r, "\mu^0_{\lambda,\lambda'}"'] \arrow[rr, bend left, "\mu^0_\lambda"] \arrow[d, "t^0_\lambda"'] & C_0(E^0_{\lambda'}) \arrow[r, "\mu^0_{\lambda'}"'] \arrow[d, "t^0_{\lambda'}"'] & C_0(E^0) \arrow[d, "t^0"'] \\
\mathcal{O}(E_\lambda) \arrow[r, "\mu_{\lambda,\lambda'}"'] \arrow[rr, bend left, "\mu_\lambda"] & \mathcal{O}(E_{\lambda'}) \arrow[r, "\mu_{\lambda'}"'] & \mathcal{O}(E) \\
C_{d_\lambda}(E^1_\lambda) \arrow[r, "\mu^1_{\lambda,\lambda'}"'] \arrow[rr, bend left, "\mu^1_\lambda"] \arrow[u, "t^1_\lambda"] & C_{d_{\lambda'}}(E^1_{\lambda'}) \arrow[r, "\mu^1_{\lambda'}"'] \arrow[u, "t^1_{\lambda'}"] & C_d(E^1) \arrow[u, "t^1"]
\end{tikzcd}
$$

We denote the natural map by $\nu_\lambda\colon \mathcal{O}(E_\lambda) \to \varinjlim \mathcal{O}(E_\lambda)$ for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, we define maps $T^0\colon C_0(E^0) \to \varinjlim \mathcal{O}(E_\lambda)$ and $T^1\colon C_d(E^1) \to \varinjlim \mathcal{O}(E_\lambda)$ so that $T^i \circ \mu^i_\lambda = \nu_\lambda \circ t^i_\lambda$ for $i = 0, 1$. By the universality of inductive limits, there exists a $*$-homomorphism $\nu\colon \varinjlim \mathcal{O}(E_\lambda) \to \mathcal{O}(E)$ with $\nu \circ \nu_\lambda = \mu_\lambda$ for $\lambda \in \Lambda$. We have $\nu \circ T^i = t^i$ for $i = 0, 1$. Hence the image of ν contains $t^0(C_0(E^0))$ and $t^1(C_d(E^1))$, and so ν is surjective. One might expect that ν is an isomorphism, as it is in the case of $\mathcal{T}(E)$. However this is not the case as the following example shows.

Example 4.9. Let $F = (F^0, F^1, d_F, r_F)$ be a discrete graph given by

$$
\begin{gathered}
F^0 = \{v, v', w\}, \qquad F^1 = \{e_k\}_{k\in\mathbb{N}}, \\
d_F(e_k) = \begin{cases} v & (k = 0) \\ v' & (k \geq 1) \end{cases}, \quad r_F(e_k) = w \ (k \in \mathbb{N}).
\end{gathered}
$$



We have $F^0_{\rm sce} = \{v, v'\}$ and $F^0_{\rm inf} = \{w\}$. Hence $F^0_{\rm rg} = \emptyset$. We see that $\mathcal{O}(F) \cong \mathbb{M}_2 \oplus \widetilde{\mathbb{K}}$ where $\widetilde{\mathbb{K}}$ means the unitization of $\mathbb{K}$. Define a regular factor map $m = (m^0, m^1)$ from F to itself by

$$
\begin{gathered}
m^0(v) = v, m^0(w) = w, m^0(v') = \infty, \\
m^1(e_0) = e_0, \ m^1(e_k) = \infty \ (k \geq 1).
\end{gathered}
$$

Under the isomorphism $\mathcal{O}(F) \cong \mathbb{M}_2 \oplus \widetilde{\mathbb{K}}$, the $*$-homomorphism $\mu\colon \mathcal{O}(F) \to \mathcal{O}(F)$ induced by m is expressed as

$$
\mathbb{M}_2 \oplus \widetilde{\mathbb{K}} \ni (x, y + \lambda) \mapsto (x, \lambda) \in \mathbb{M}_2 \oplus \widetilde{\mathbb{K}}
$$

for $x \in \mathbb{M}_2$, $y \in \mathbb{K}$ and $\lambda \in \mathbb{C}$. For $k \in \mathbb{N}$, define $E_k = F$ and $m_{k,k+1} = m$. Then $\{E_k\}_{k\in\mathbb{N}}$ and $\{m_{k,k+1}\}_{k\in\mathbb{N}}$ give a regular projective system. Its projective limit $E = (E^0, E^1, d, r)$ is a discrete graph such that $E^0 = \{v, w\}, E^1 = \{e_0\}, d(e_0) = v, r(e_0) = w$. We have $\mathcal{O}(E) \cong \mathbb{M}_2$. On the other hand, we have $\varinjlim \mathcal{O}(E_k) \cong \mathbb{M}_2 \oplus \mathbb{C}$. Hence the surjection $\varinjlim \mathcal{O}(E_k) \to \mathcal{O}(E)$ is not an isomorphism. Note that $\varinjlim \mathcal{O}(E_k) \cong \mathcal{T}(E)$.

We define an open set $O \subset E^0$ by

$$
O = \bigcup_{\lambda\in\Lambda} (m^0_\lambda)^{-1}((E^0_\lambda)_{\rm rg}).
$$

From Lemma 2.7 and Proposition 4.8, we see that $(m^0_\lambda)^{-1}((E^0_\lambda)_{\rm rg}) \subset E^0_{\rm rg}$ for every $\lambda \in \Lambda$. Hence O is an open subset of $E^0_{\rm rg}$.

Lemma 4.10. *We have* $\{f \in C_0(E^0_{\rm rg}) \mid T^0(f) = \Phi(\pi_r(f))\} = C_0(O)$.

Proof. Take $\lambda \in \Lambda$ and $g \in C_0((E^0_\lambda)_{\mathrm{rg}})$. We have $\mu^0_\lambda(g) \in C_0(O) \subset C_0(E^0_{\mathrm{rg}})$. We define $\psi_\lambda\colon \mathcal{K}(C_{d_\lambda}(E^1_\lambda)) \to \mathcal{K}(C_d(E^1))$ by $\psi_\lambda(\theta_{\xi,\eta}) = \theta_{\mu^1_\lambda(\xi),\mu^1_\lambda(\eta)}$ for $\xi,\eta \in C_{d_\lambda}(E^1_\lambda)$. Then we have $\Phi \circ \psi_\lambda = \nu_\lambda \circ \varphi_\lambda$. By Lemma 2.8, we have

$$\begin{aligned} T^0(\mu^0_\lambda(g)) &= \nu_\lambda(t^0_\lambda(g)) = \nu_\lambda(\varphi_\lambda(\pi_{r_\lambda}(g))) \\ &= \Phi(\psi_\lambda(\pi_{r_\lambda}(g))) = \Phi(\pi_r(\mu^0_\lambda(g))). \end{aligned}$$

Since

$$C_0(O) = \overline{\bigcup_{\lambda\in\Lambda} \mu^0_\lambda\big(C_0((E^0_\lambda)_{\mathrm{rg}})\big)},$$

we have $T^0(f) = \Phi(\pi_r(f))$ for all $f \in C_0(O)$.

To derive a contradiction, assume that there exists $f \in C_0(E^0_{\mathrm{rg}})$ such that $T^0(f) = \Phi(\pi_r(f))$ and $f \notin C_0(O)$. There exists $v \notin O$ with $|f(v)| = \varepsilon > 0$. We can find $\lambda_0 \in \Lambda$ and $g_0 \in C_0(E^0_\lambda)$ such that $\|\nu_{\lambda_0}(t^0_{\lambda_0}(g_0)) - T^0(f)\| < \varepsilon/3$. Since

$$\Phi\big(\mathcal{K}(C_d(E^1))\big) = \overline{\bigcup_{\lambda\in\Lambda} \nu_\lambda\big(\varphi_\lambda\big(\mathcal{K}(C_{d_\lambda}(E^0_\lambda))\big)\big)},$$

we can find $\lambda_1 \in \Lambda$ and $x_1 \in \mathcal{K}(C_{d_{\lambda_1}}(E^0_{\lambda_1}))$ such that $\|\nu_{\lambda_1}(\varphi_{\lambda_1}(x_1)) - \Phi(\pi_r(f))\| < \varepsilon/3$. Take $\lambda_2 \in \Lambda$ so that $\lambda_2 \succeq \lambda_0$ and $\lambda_2 \succeq \lambda_1$, and set

$$a = \mu_{\lambda_0,\lambda_2}\big(t^0_{\lambda_0}(g_0)\big) - \mu_{\lambda_1,\lambda_2}\big(\varphi_{\lambda_1}(x_1)\big) \in \mathcal{O}(E_{\lambda_2}).$$

Then we have

$$\begin{aligned} \|\nu_{\lambda_2}(a)\| &= \big\|\nu_{\lambda_0}\big(t^0_{\lambda_0}(g_0)\big) - \nu_{\lambda_1}\big(\varphi_{\lambda_1}(x_1)\big)\big\| \\ &\le \big\|\nu_{\lambda_0}\big(t^0_{\lambda_0}(g_0)\big) - T^0(f)\big\| + \big\|\Phi\big(\pi_r(f)\big) - \nu_{\lambda_1}\big(\varphi_{\lambda_1}(x_1)\big)\big\| \\ &< 2\varepsilon/3. \end{aligned}$$

By the definition of the inductive limit of C^*-algebras, there exists $\lambda_3 \succeq \lambda_2$ such that $\|\mu_{\lambda_2,\lambda_3}(a)\| < 2\varepsilon/3$. Take $\lambda \in \Lambda$ such that $\lambda \succeq \lambda_3$ and $m^0_\lambda(v) \in E^0_\lambda$. Set $g = \mu^0_{\lambda_0,\lambda}(g_0) \in C_0(E^0_\lambda)$ and $x = \psi_{\lambda_1,\lambda}(x_1) \in \mathcal{K}(C_{d_\lambda}(E^1_\lambda))$, where $\psi_{\lambda_1,\lambda}\colon \mathcal{K}(C_{d_{\lambda_1}}(E^1_{\lambda_1})) \to \mathcal{K}(C_{d_\lambda}(E^1_\lambda))$ is defined similarly as ψ_λ. Then we have

$$\begin{aligned} \|t^0_\lambda(g) - \varphi(x)\| &= \big\|\mu_{\lambda_0,\lambda}\big(t^0_{\lambda_0}(g_0)\big) - \mu_{\lambda_1,\lambda}\big(\varphi_{\lambda_1}(x_1)\big)\big\| \\ &= \big\|\mu_{\lambda_2,\lambda}\big(\mu_{\lambda_0,\lambda_2}(t^0_{\lambda_0}(g_0)) - \mu_{\lambda_1,\lambda_2}(\varphi_{\lambda_1}(x_1))\big)\big\| \\ &= \|\mu_{\lambda_2,\lambda}(a)\| = \big\|\mu_{\lambda_3,\lambda}\big(\mu_{\lambda_2,\lambda_3}(a)\big)\big\| \le \|\mu_{\lambda_2,\lambda_3}(a)\| < 2\varepsilon/3. \end{aligned}$$

Since $v \notin O$, we have $m^0_\lambda(v) \in (E^0_\lambda)_{\mathrm{sg}}$. Since

$$\|\mu^0_\lambda(g) - f\| = \|T^0(\mu^0_{\lambda_0}(g_0) - f)\| = \|\nu_{\lambda_0}(t^0_{\lambda_0}(g_0)) - T^0(f)\| < \varepsilon/3,$$

we have $|(\mu^0_\lambda(g) - f)(v)| < \varepsilon/3$. Hence we have $|g(m^0_\lambda(v))| > 2\varepsilon/3$ because $|f(v)| = \varepsilon$. Since $\mathcal{F}^1/\mathcal{G}^1 \cong C_0((E^0_\lambda)_{sg})$, we have

$$\inf_{x' \in \mathcal{K}(C_{d_\lambda}(E^1_\lambda))} \|t^0_\lambda(g) - \varphi_\lambda(x')\| = \sup_{v' \in (E^0_\lambda)_{sg}} |g(v')|.$$

However, we have $\|t^0_\lambda(g) - \varphi_\lambda(x)\| < 2\varepsilon/3$ and $|g(m^0_\lambda(v))| > 2\varepsilon/3$. This is a contradiction. Thus we have shown that $T^0(f) = \Phi(\pi_r(f))$ if and only if $f \in C_0(O)$. □

We define $Y = E^0_{\mathrm{rg}} \setminus O$ which is a closed subset of E^0_{rg}. In Example 4.9, we have $O = \emptyset$ and $E^0_{\mathrm{rg}} = \{w\}$, hence $Y = \{w\}$. It is easy to see the following.

Proposition 4.11. *The inductive limit* $\varinjlim \mathcal{O}(E_\lambda)$ *is isomorphic to* $\mathcal{O}(E_Y)$*, and the surjection* $\nu\colon \varinjlim \mathcal{O}(E_\lambda) \to \mathcal{O}(E)$ *is an isomorphism if and only if* $Y = \emptyset$*.*

Let us say that a projective system $(\{E_\lambda\}_{\lambda\in\Lambda}, \{m_{\lambda,\lambda'}\}_{\lambda\preceq\lambda'})$ is surjective when $m^0_{\lambda,\lambda'}$ is surjective for every $\lambda \preceq \lambda'$. By Proposition 2.9, we get an injective inductive system of C^*-algebras from a surjective regular projective system.

Lemma 4.12. *If a regular projective system* $(\{E_\lambda\}_{\lambda\in\Lambda}, \{m_{\lambda,\lambda'}\}_{\lambda\preceq\lambda'})$ *is surjective, then we have* $Y = \emptyset$*.*

Proof. Suppose that $m^0_{\lambda,\lambda'}$ is surjective for every $\lambda \preceq \lambda'$. Note that $m^1_{\lambda,\lambda'}$ is surjective for every $\lambda \preceq \lambda'$ and that m^0_λ, m^1_λ is surjective for every $\lambda \in \Lambda$. To prove that $Y = \emptyset$, it suffices to see that for $v \in E^0_{\mathrm{rg}}$, there exists $\lambda \in \Lambda$ such that $m^0_\lambda(v) \in (E^0_\lambda)_{\mathrm{rg}}$. Take $v \in E^0_{\mathrm{rg}}$. By Lemma 1.4, there exists a compact neighborhood V of v such that $r^{-1}(V)$ is compact and $r(r^{-1}(V)) = V$. Since V is a neighborhood of v, there exist $\lambda_0 \in \Lambda$ and a neighborhood V' of $m^0_{\lambda_0}(v)$ such that $(m^0_{\lambda_0})^{-1}(V') \subset V$. For $\lambda \succeq \lambda_0$, we have $(m^0_{\lambda_0,\lambda})^{-1}(V') \subset m^0_\lambda(V)$ because m^0_λ is surjective. Hence $m^0_\lambda(V)$ is a neighborhood of $m^0_\lambda(v)$ for $\lambda \succeq \lambda_0$. Since $\widetilde{E}^0 \setminus V$ is a neighborhood of $\infty \in \widetilde{E}^0$, there exists $\lambda_1 \in \Lambda$ such that $(m^0_{\lambda_1})^{-1}(V'') \subset \widetilde{E}^0 \setminus V$ for some neighborhood V'' of $\infty \in \widetilde{E}^0_{\lambda_1}$. Hence $\infty \notin m_{\lambda_1}(V)$. This implies that $m^0_\lambda(V) \subset E^0_\lambda$ for every $\lambda \succeq \lambda_1$. Similarly there exists $\lambda_2 \in \Lambda$ such that $m^1_\lambda(r^{-1}(V)) \subset E^1_\lambda$ for every $\lambda \succeq \lambda_2$ because $r^{-1}(V) \subset E^1$ is compact. Take $\lambda \in \Lambda$ with $\lambda \succeq \lambda_i$ for $i = 0, 1, 2$. Set $V_\lambda = m^0_\lambda(V)$ and $U_\lambda = m^1_\lambda(r^{-1}(V))$. Then V_λ is a compact neighborhood of $m^0_\lambda(v)$ in E^0_λ and U_λ is a compact subset of E^1_λ. We get $U_\lambda = r_\lambda^{-1}(V_\lambda)$ and $r_\lambda(U_\lambda) = V_\lambda$ because m^0_λ and m^1_λ are surjective and $r(r^{-1}(V)) = V$. Hence we have $m^0_\lambda(v) \in (E^0_\lambda)_{rg}$ by Lemma 1.4. Thus we have shown that $Y = \emptyset$. ∎

By Proposition 4.11 and Lemma 4.12, we get the following which is satisfactory for application.

Proposition 4.13. *For a surjective regular projective system* $(\{E_\lambda\}_{\lambda\in\Lambda}, \{m_{\lambda,\lambda'}\}_{\lambda\preceq\lambda'})$*, we have* $\mathcal{O}\big(\varprojlim E_\lambda\big) \cong \varinjlim \mathcal{O}(E_\lambda)$*.*

5. Subalgebras of $\mathcal{O}(E)$

Let us take a topological graph $E = (E^0, E^1, d, r)$ and fix it. In this section, we study subalgebras of $\mathcal{O}(E)$.

Definition 5.1. A *subgraph* of the topological graph $E = (E^0, E^1, d, r)$ is a quadruple $F = (F^0, F^1, d_F, r_F)$ where $F^0 \subset E^0$ and $F^1 \subset E^1$ are open subsets such that $d(F^1), r(F^1) \subset F^0$, and d_F, r_F are the restrictions of d, r to F^1.

Take a subgraph $F = (F^0, F^1, d_F, r_F)$ of the topological graph $E = (E^0, E^1, d, r)$. We can identify $C_{d_F}(F^1)$ with $C_d(E^1) \cap C_0(F^1)$. The C^*-algebra generated by $t^0(C_0(F^0))$ and $t^1(C_{d_F}(F^1))$ is different from $\mathcal{O}(F)$ in general. We will explore what the difference is. Let us define a $*$-homomorphism $T^0\colon C_0(F^0) \to \mathcal{O}(E)$ and a linear map $T^1\colon C_{d_F}(F^1) \to \mathcal{O}(E)$ by the restrictions of t^0 and t^1, respectively. It is clear that the pair $T = (T^0, T^1)$ is an injective Toeplitz F-pair. In general, it is not a Cuntz-Krieger F-pair.

Lemma 5.2. *For $f \in C_0(F^0)$, we have $T^0(f) = \Phi(\pi_{r_F}(f))$ if and only if $f \in C_0(F^0_{\mathrm{rg}} \setminus \overline{r(E^1 \setminus F^1)})$.*

Proof. First note that the $*$-homomorphism $\Phi\colon \mathcal{K}(C_{d_F}(F^1)) \to \mathcal{O}(E)$ obtained by the Toeplitz F-pair T is the restriction of the map $\varphi\colon \mathcal{K}(C_d(E^1)) \to \mathcal{O}(E)$ to $\mathcal{K}(C_{d_F}(F^1)) \subset \mathcal{K}(C_d(E^1))$. We first show that $f \in C_0(F^0)$ satisfies $T^0(f) = \Phi(\pi_{r_F}(f))$ if and only if $f \in C_0(F^0 \cap E^0_{\mathrm{rg}} \setminus \overline{r(E^1 \setminus F^1)})$. Let us take $f \in C_0(F^0)$ satisfies $T^0(f) = \Phi(\pi_{r_F}(f))$. Since we have $t^0(f) \in \varphi(\mathcal{K}(C_d(E^1)))$, we get $f \in C_0(E^0_{\mathrm{rg}})$ and

$$\Phi(\pi_{r_F}(f)) = T^0(f) = t^0(f) = \varphi(\pi_r(f)).$$

The latter condition implies that $f \circ r \in C_0(F^1)$ and so $f \in C_0(E^0 \setminus \overline{r(E^1 \setminus F^1)})$. Therefore $f \in C_0(F^0 \cap E^0_{\mathrm{rg}} \setminus \overline{r(E^1 \setminus F^1)})$. Conversely if $f \in C_0(F^0)$ satisfies $f \in C_0(F^0 \cap E^0_{\mathrm{rg}} \setminus \overline{r(E^1 \setminus F^1)})$, then we get

$$T^0(f) = t^0(f) = \varphi(\pi_r(f)) = \Phi(\pi_{r_F}(f)).$$

Thus we have shown that $f \in C_0(F^0)$ satisfies $T^0(f) = \Phi(\pi_{r_F}(f))$ if and only if $f \in C_0\big(F^0 \cap E^0_{\mathrm{rg}} \setminus \overline{r(E^1 \setminus F^1)}\big)$. The proof completes once we show

$$F^0 \cap E^0_{\mathrm{rg}} \setminus \overline{r(E^1 \setminus F^1)} = F^0_{\mathrm{rg}} \setminus \overline{r(E^1 \setminus F^1)}.$$

Take $v \in F^0_{\mathrm{rg}} \setminus \overline{r(E^1 \setminus F^1)}$. By Proposition 1.4 we can find a neighborhood $V \subset F^0$ of $v \in F^0$ satisfying the conditions that $(r_F)^{-1}(V) \subset F^1$ is compact and $r_F((r_F)^{-1}(V)) = V$. By replacing it by a smaller set if necessary, we may assume that $V \cap r(E^1 \setminus F^1) = \emptyset$. Since F^0 is an open subset of E^0, V is a neighborhood of $v \in E^0$. Since $V \cap r(E^1 \setminus F^1) = \emptyset$, we have $r^{-1}(V) \subset F^1$. Hence $r^{-1}(V) = (r_F)^{-1}(V)$ is compact and satisfies $r(r^{-1}(V)) = V$. By Proposition 1.4, we have $v \in E^0_{\mathrm{rg}}$. Thus we get $v \in F^0 \cap E^0_{\mathrm{rg}} \setminus \overline{r(E^1 \setminus F^1)}$. Conversely, take $v \in F^0 \cap E^0_{\mathrm{rg}} \setminus \overline{r(E^1 \setminus F^1)}$. Then we can find a neighborhood V of $v \in E^0$ such that $r^{-1}(V)$ is compact and $r(r^{-1}(V)) = V$. By replacing it by a smaller set if necessary, we may assume $V \subset F^0$ and $V \cap r(E^1 \setminus F^1) = \emptyset$. Then $V \subset F^0$ is a neighborhood of $v \in F^0$ such that $(r_F)^{-1}(V) \subset F^1$ is compact and $r_F((r_F)^{-1}(V)) = V$. Thus we have $v \in F^0_{\mathrm{rg}} \setminus \overline{r(E^1 \setminus F^1)}$. Therefore we get

$$F^0 \cap E^0_{\mathrm{rg}} \setminus \overline{r(E^1 \setminus F^1)} = F^0_{\mathrm{rg}} \setminus \overline{r(E^1 \setminus F^1)}.$$

This completes the proof. □

Proposition 5.3. *For a subgraph $F = (F^0, F^1, d_F, r_F)$ of the topological graph E, the C^*-subalgebra of $\mathcal{O}(E)$ generated by $t^0(C_0(F^0))$ and $t^1(C_{d_F}(F^1))$ is isomorphic to $\mathcal{O}(F_Y)$ where Y is the closed subset of F^0_{rg} defined by $Y = F^0_{\mathrm{rg}} \cap \overline{r(E^1 \setminus F^1)}$.*

Proof. The injective Toeplitz F-pair $T = (T^0, T^1)$ admits a gauge action and $C^*(T)$ is the C^*-subalgebra of $\mathcal{O}(E)$ generated by $t^0(C_0(F^0))$ and $t^1(C_{d_F}(F^1))$. By Lemma 5.2, we have $Y_T = Y$. Hence by Corollary 3.19, the C^*-subalgebra of $\mathcal{O}(E)$ generated by $t^0(C_0(F^0))$ and $t^1(C_{d_F}(F^1))$ is isomorphic to $\mathcal{O}(F_Y)$. □

If a subgraph F of E satisfies $F^0_{\mathrm{rg}} \cap r(E^1 \setminus F^1) = \emptyset$, then the C^*-subalgebra of $\mathcal{O}(E)$ generated by $t^0(C_0(F^0))$ and $t^1(C_{d_F}(F^1))$ is isomorphic to $\mathcal{O}(F)$ by Proposition 5.3. The following is one useful construction of such subgraphs.

For an open subset V of E^0, we define $F_V^0 = V \cup d(r^{-1}(V)) \subset E^0$ and $F_V^1 = r^{-1}(V) \subset E^1$. We see that F_V^0, F_V^1 are open subsets, and that $d(F_V^1), r(F_V^1) \subset F_V^0$. Hence $F_V = (F_V^0, F_V^1, d_V, r_V)$ is a subgraph of the topological graph E, where d_V, r_V are the restrictions of d, r to F_V^1. We denote by $A_V \subset \mathcal{O}(E)$ the C^*-subalgebra of $\mathcal{O}(E)$ generated by $t^0(C_0(F_V^0))$ and $t^1(C_{d_V}(F_V^1))$. We will show that $A_V \cong \mathcal{O}(F_V)$.

Lemma 5.4. *We have $(F_V^0)_{\mathrm{rg}} = V \cap E_{\mathrm{rg}}^0$.*

Proof. Take $v \in (F_V^0)_{rg}$. By Lemma 1.4, there exists a neighborhood W of $v \in F_V^0$ such that $r_V^{-1}(W) \subset F_V^1$ is compact and $r_V(r_V^{-1}(W)) = W$. We can choose such a W with $W \subset V$ because $v \in (F_V^0)_{\mathrm{rg}} \subset r(F_V^1) \subset V$. Hence we have $r_V^{-1}(W) = r^{-1}(W)$. This implies that $v \in E_{\mathrm{rg}}^0$ by Lemma 1.4. Thus $(F_V^0)_{\mathrm{rg}} \subset V \cap E_{\mathrm{rg}}^0$.

Conversely take $v \in V \cap E_{\mathrm{rg}}^0$. Then we can find a neighborhood W of $v \in E^0$ such that $W \subset V$, $r^{-1}(W) \subset E^1$ is compact and $r(r^{-1}(W)) = W$ by Lemma 1.4. This implies that $v \in (F_V^0)_{\mathrm{rg}}$. Hence $(F_V^0)_{\mathrm{rg}} \supset V \cap E_{\mathrm{rg}}^0$. Therefore we have $(F_V^0)_{\mathrm{rg}} = V \cap E_{\mathrm{rg}}^0$. □

Proposition 5.5. *For an open subset V of E^0, we have $A_V \cong \mathcal{O}(F_V)$.*

Proof. By Proposition 5.3, it suffices to check $(F_V^0)_{\mathrm{rg}} \cap r(E^1 \setminus F^1) = \emptyset$, which easily follows from Lemma 5.4. □

Proposition 5.6. *If an increasing family of open subsets $V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$ of E^0 satisfies $\bigcup_{k=1}^\infty V_k = E^0$, then we have $\mathcal{O}(E) = \overline{\bigcup_{k=1}^\infty A_{V_k}}$ and $\mathcal{O}(E) \cong \lim_{k\to\infty} \mathcal{O}(F_{V_k})$.*

Proof. It is easy to see that we have $A_{V_1} \subset A_{V_2}$ for two open subsets V_1, V_2 of E^0 satisfying $V_1 \subset V_2$. Hence $\overline{\bigcup_{k=1}^\infty A_{V_k}}$ is a C^*-subalgebra of $\mathcal{O}(E)$ which contains $t^0(C_0(E^0))$ and $t^1(C_d(E^1))$ because $\bigcup_{k=1}^\infty V_k = E^0$. Hence we have $\overline{\bigcup_{k=1}^\infty A_{V_k}} = \mathcal{O}(E)$. By Proposition 5.5, we get $\mathcal{O}(E) \cong \lim_{k\to\infty} \mathcal{O}(F_{V_k})$. □

Remark 5.7. For two open subsets V_1, V_2 of E^0 satisfying $V_1 \subset V_2$, the injective map $\mathcal{O}(F_{V_1}) \to \mathcal{O}(F_{V_2})$ induced by the inclusion $A_{V_1} \subset A_{V_2}$ is the same map as the one induced by a regular factor map $m = (m^0, m^1)$ from F_{V_2} to F_{V_1} defined by $m^i|_{F_{V_1}^i} = \mathrm{id}_{F_{V_1}^i}$ and $m^i(F_{V_2}^i \setminus F_{V_1}^i) = \infty$ for $i = 0, 1$. Thus the latter statement of Proposition 5.6 is the special case of Proposition 4.13.

We determine conditions on an open subset V that imply A_V is hereditary or full.

Lemma 5.8. *For an open subset V of E^0 with $d(r^{-1}(V)) \subset V$, we have $F_V^n = (r^n)^{-1}(V)$ for $n \in \mathbb{N}$.*

Proof. For $n = 0$, we have $F_V^0 = V \cup d(r^{-1}(V)) = V$. For $n = 1$, by definition $F_V^1 = r^{-1}(V)$. Let n be an integer greater than 1. Since $F_V^1 = r^{-1}(V)$, we have $F_V^n \subset (r^n)^{-1}(V)$. Take $e = (e_1, e_2, \ldots, e_n) \in (r^n)^{-1}(V)$. From $r(e_1) = r^n(e) \in V$, we get $e_1 \in r^{-1}(V) = F_V^1$. Then we have $r(e_2) = d(e_1) \in d(r^{-1}(V)) \subset V$. Hence we get $e_2 \in r^{-1}(V) = F_V^1$. Recursively, we have $e_i \in r^{-1}(V) = F_V^1$ for $i = 1, 2, \ldots, n$. Hence we get $e \in F_V^n$. Therefore we have $F_V^n = (r^n)^{-1}(V)$ for $n \in \mathbb{N}$. □

Proposition 5.9. *If an open subset V of E^0 satisfies $d(r^{-1}(V)) \subset V$, then A_V is a hereditary subalgebra of $\mathcal{O}(E)$. If in addition each $v \in E^0 \setminus V$ is regular and satisfies $d^n((r^n)^{-1}(v)) \subset V$ for some $n \in \mathbb{N}$, then the hereditary subalgebra A_V is full in $\mathcal{O}(E)$. Hence in this case, $\mathcal{O}(F_V)$ is strongly Morita equivalent to $\mathcal{O}(E)$.*

Proof. Let V be an open subset of E^0 such that $d(r^{-1}(V)) \subset V$. The linear span of elements of the form $t^n(\xi)t^m(\eta)^*$ is dense in $\mathcal{O}(E)$ (see [K1, Section 2] for the proof of this fact and the definition of the linear map $t^n \colon C_d(E^n) \to \mathcal{O}(E)$). Therefore, once we get $t^0(f)t^n(\xi) \in A_V$ for arbitrary $f \in C_0(V)$, $\xi \in C_d(E^n)$ and $n \in \mathbb{N}$. we know that A_V is a hereditary subalgebra generated by $t^0(C_0(V))$. Take $f \in C_0(V)$ and $\xi \in C_d(E^n)$ for $n \in \mathbb{N}$. For $e \notin F_V^n$, we have $(\pi_{r^n}(f)\xi)(e) = f(r^n(e))\xi(e) = 0$ because $r^n(e) \notin V$ by Lemma 5.8. Hence we have $\pi_{r^n}(f)\xi \in C_{d_V}(F_V^n)$. Thus we get

$$t^0(f)t^n(\xi) = t^n(\pi_{r^n}(f)\xi) \in t^n(C_{d_V}(F_V^n)) \subset A_V.$$

This proves the first part.

Now further assume that each $v \in E^0 \setminus V$ is regular and satisfies $d^n((r^n)^{-1}(v)) \subset V$ for some $n \in \mathbb{N}$. We set $V_0 = V$, and define $V_n \subset E^0$ for $n = 1, 2, \ldots$ by

$$V_n = \{v \in E^0 \mid d^n((r^n)^{-1}(v)) \subset V\}.$$

Since V satisfies $d(r^{-1}(V)) \subset V$, we have $V_n \subset V_{n+1}$ for $n \in \mathbb{N}$. By the assumption, we have $\bigcup_{n=0}^{\infty} V_n = E^0$. For $n \in \mathbb{N}$, we have

$$V_{n+1} = \{v \in E^0 \mid r^{-1}(v) \subset d^{-1}(V_n)\}.$$

If we set

$$V'_{n+1} = V_{n+1} \cap E^0_{\mathrm{rg}} = \{v \in E^0_{\mathrm{rg}} \mid r^{-1}(v) \subset d^{-1}(V_n)\},$$

then we have $V_{n+1} = V \cup V'_{n+1}$ because $E^0_{\mathrm{sg}} \subset V$. By [K1, Lemma 1.21], if V_n is open then V'_{n+1} is open. Hence we can show that V_n is open recursively. Let I be an ideal generated by A_V. We will show $t^0(C_0(V_n)) \subset I$ by induction with respect to $n \in \mathbb{N}$. For $n = 0$, we have $t^0(C_0(V_0)) = t^0(C_0(V)) \subset A_V \subset I$. Assume we have $t^0(C_0(V_n)) \subset I$ for $n \in \mathbb{N}$. Take $f \in C_0(V'_{n+1})$. Since $V'_{n+1} \subset E^0_{\mathrm{rg}}$, we have $t^0(f) = \varphi(\pi_r(f))$. By $r^{-1}(V'_{n+1}) \subset d^{-1}(V_n)$, we see $\pi_r(f) \in C_0(d^{-1}(V_n)) \subset \mathcal{K}(C_d(d^{-1}(V_n)))$. Since $t^0(C_0(V_n)) \subset I$, we have $t^1(C_d(d^{-1}(V_n))) \subset I$. Hence $\varphi\big(\mathcal{K}(C_d(d^{-1}(V_n)))\big) \subset I$. This shows that $t^0(f) \in I$. Hence we have $t^0(C_0(V_{n+1})) \subset I$ because $V_{n+1} = V \cup V'_{n+1}$. Thus we have shown that $t^0(C_0(V_n)) \subset I$ for all $n \in \mathbb{N}$. Since $\bigcup_{n=0}^{\infty} V_n = E^0$, we get $t^0(C_0(E^0)) \subset I$. Since $\mathcal{O}(E)$ is generated by $t^0(C_0(E^0))$ as a hereditary subalgebra, we have $I = \mathcal{O}(E)$. Thus A_V is a full hereditary subalgebra.

The last part follows from Proposition 5.5. □

Remark 5.10. We will see in [K5, Remark 6.2] that for an open subset V of E^0 with $d(r^{-1}(V)) \subset V$, the condition that each $v \in E^0 \setminus V$ is regular and satisfies $d^n((r^n)^{-1}(v)) \subset V$ for some $n \in \mathbb{N}$ is not only sufficient but also necessary for A_V to be full.

6. Strong Morita equivalence

In Proposition 5.9, we found subgraphs F of E such that $\mathcal{O}(F)$ is strongly Morita equivalent to $\mathcal{O}(E)$. In this section, we give a construction of a topological graph F which contains a given topological graph E such that $\mathcal{O}(F)$ is strongly Morita equivalent to $\mathcal{O}(E)$.

Let $E = (E^0, E^1, d, r)$ be a topological graph, and N be a positive integer or ∞. For $k = 1, 2, \ldots, N$, take locally compact spaces X_k^0, X_k^1, local homeomorphisms

$d_k\colon X_k^1 \to X_{k-1}^0$, and proper continuous surjections $r_k\colon X_k^1 \to X_k^0$, where $X_0^0 = E^0$. Set

$$F^0 = E^0 \amalg X_1^0 \amalg \cdots \amalg X_N^0, \quad F^1 = E^1 \amalg X_1^1 \amalg \cdots \amalg X_N^1$$

and define $d_F, r_F\colon F^1 \to F^0$ from d, r and d_k, r_k for $k = 1, 2, \ldots, N$. Then $F = (F^0, F^1, d_F, r_F)$ is a topological graph.

Lemma 6.1. *We have $F^0_{\mathrm{rg}} = E^0_{\mathrm{rg}} \amalg X_1^0 \amalg \cdots \amalg X_N^0$.*

Proof. Clearly, $F^0_{\mathrm{rg}} \cap E^0 = E^0_{\mathrm{rg}}$. For $k = 1, 2, \ldots, N$, we have $X_k^0 \subset F^0_{\mathrm{rg}}$ because r_k is surjective and proper. Hence we have $F^0_{\mathrm{rg}} = E^0_{\mathrm{rg}} \amalg X_1^0 \amalg \cdots \amalg X_N^0$. □

We have

$$C_0(F^0) = C_0(E^0) \oplus C_0(X_1^0) \oplus \cdots \oplus C_0(X_N^0),$$
$$C_{d_F}(F^0) = C_d(E^1) \oplus C_{d_1}(X_1^1) \oplus \cdots \oplus C_{d_N}(X_N^1).$$

Hence there exist natural inclusions $\mu^0\colon C_0(E^0) \to C_0(F^0)$ and $\mu^1\colon C_d(E^1) \to C_{d_F}(F^1)$. Let $t_E = (t_E^0, t_E^1)$ and $t_F = (t_F^0, t_F^1)$ be the universal Cuntz Krieger E-pair in $\mathcal{O}(E)$ and the universal Cuntz Krieger F-pair in $\mathcal{O}(F)$, respectively.

Proposition 6.2. *There exists an injective $*$-homomorphism $\mu\colon \mathcal{O}(E) \to \mathcal{O}(F)$ such that $t_F^i \circ \mu^i = \mu \circ t_E^i$ for $i = 0, 1$.*

Proof. It is routine to check that the pair $T = (t_F^0 \circ \mu^0, t_F^1 \circ \mu^1)$ is a Toeplitz E-pair. By Lemma 6.1, T is a Cuntz-Krieger E-pair. Hence there exists a $*$-homomorphism $\mu\colon \mathcal{O}(E) \to \mathcal{O}(F)$ such that $t_F^i \circ \mu^i = \mu \circ t_E^i$ for $i = 0, 1$. Clearly the pair T is injective and admits a gauge action. Hence μ is injective by Proposition 1.6, □

Remark 6.3. As in Remark 5.7, we see that the injection $\mu\colon \mathcal{O}(E) \to \mathcal{O}(F)$ in Proposition 6.2 is induced by a regular factor map from F to E.

Proposition 6.4. *The image $\mu(\mathcal{O}(E))$ is a full corner of $\mathcal{O}(F)$. Hence $\mathcal{O}(F)$ is strongly Morita equivalent to $\mathcal{O}(E)$.*

Proof. Since $t_F^0\colon C_0(F^0) \to \mathcal{O}(F)$ is non-degenerate by [K1, Lemma 1.20], it extends the map $C_b(F^0) = \mathcal{M}(C_0(F^0)) \to \mathcal{M}(\mathcal{O}(F))$ where $\mathcal{M}(\cdot)$ means a multiplier algebra (see [La, Proposition 2.1]). Let $p \in \mathcal{M}(\mathcal{O}(F))$ be the image of the characteristic function of $E^0 \subset F^0$ under this map. We will show that $p\mathcal{O}(F)p = \mu(\mathcal{O}(E))$. Since $\mathcal{O}(F)$ is the linear span of elements of the form $t_F^n(\xi)t_F^m(\eta)^*$ for $\xi \in C_{d_F}(F^n)$, $\eta \in C_{d_F}(F^m)$ and $n, m \in \mathbb{N}$, the corner $p\mathcal{O}(F)p$ is the linear span of elements of the form $pt_F^n(\xi)t_F^m(\eta)^*p$ (see [K1, Section 2]). For $\xi \in C_{d_F}(F^n)$, we have $pt_F^n(\xi) = \mu(t_E^n(\xi_0))$ where $\xi_0 \in C_d(E^n)$ is the restriction of ξ to $E^n \subset F^n$, because for $e \in F^n$, $r_F^n(e) \in E^0$ implies $e \in E^n$. Thus we have $p\mathcal{O}(F)p = \mu(\mathcal{O}(E))$.

Let I be an ideal of $\mathcal{O}(F)$ generated by $p\mathcal{O}(F)p$. We will prove that $I = \mathcal{O}(F)$ in a fashion that is similar to the proof of Proposition 5.9. By the former part, we have $t_F^0(C_0(E^0)), t_F^1(C_d(E^1)) \subset I$. We will prove $t_F^0(C_0(X_1^0)), t_F^1(C_{d_1}(X_1^1)) \subset I$. Since $d_1(X_1^1) \subset E^0$, we see that $t_F^1(\xi)p = t_F^1(\xi)$ for every $\xi \in C_{d_1}(X_1^1) \subset C_{d_F}(F^1)$. Therefore we get $t_F^1(\xi) \in I$ for every $\xi \in C_{d_1}(X_1^1)$. This implies that $\varphi_F\big(\mathcal{K}(C_{d_1}(X_1^1))\big) \subset I$. For $f \in C_0(X_1^0)$, we have $\pi_{r_F}(f) \in \mathcal{K}(C_{d_1}(X_1^1))$, and so $t_F^0(f) = \varphi_F(\pi_{r_F}(f)) \in I$ because $X_1^0 \subset F^0_{\mathrm{rg}}$. Thus we get $t_F^0(C_0(X_1^0)), t_F^1(C_{d_1}(X_1^1)) \subset I$. The same argument shows that $t_F^0(C_0(X_k^0)), t_F^1(C_{d_k}(X_k^1)) \subset I$ for $k = 2, \ldots, N$

by induction. Hence we get $t_F^0(C_0(F^0)), t_F^1(C_{d_F}(F^1)) \subset I$. Since $\mathcal{O}(F)$ is generated by the images of t_F^0 and t_F^1, we have $I = \mathcal{O}(F)$. Therefore $\mu(\mathcal{O}(E))$ is a full corner of $\mathcal{O}(F)$. The last part follows from the injectivity of μ. $\square$

Now we specialize the above discussion. Let $E = (E^0, E^1, d, r)$ be a topological graph and N be a positive integer. For $k = 1, 2, \ldots, N$, set $X_k^0, X_k^1 \cong E^0$, and define $d_k\colon X_k^1 \to X_{k-1}^0$ and $r_k\colon X_k^1 \to X_k^0$ by the identity map of E^0 where $X_0^0 = E^0$. Set

$$E_N^0 = E^0 \amalg X_1^0 \amalg \cdots \amalg X_N^0, \quad E_N^1 = E^1 \amalg X_1^1 \amalg \cdots \amalg X_N^1$$

and define $d_N, r_N\colon E_N^1 \to E_N^0$ from d, r and d_k, r_k $(k = 1, 2, \ldots, N)$. Then $E_N = (E_N^0, E_N^1, d_N, r_N)$ is a topological graph. By Proposition 6.4, the C^*-algebra $\mathcal{O}(E_N)$ is strongly Morita equivalent to $\mathcal{O}(E)$. We can say more.

Proposition 6.5. *We have* $\mathcal{O}(E_N) \cong \mathcal{O}(E) \otimes \mathbb{M}_{N+1}$.

Proof. In the proof, we use the natural identification

$$\begin{aligned} C_0(E_N^0) &= C_0(E^0) \oplus C_0(X_1^0) \oplus \cdots \oplus C_0(X_N^0), \\ C_{d_N}(E_N^1) &= C_d(E^1) \oplus C_{d_1}(X_1^1) \oplus \cdots \oplus C_{d_N}(X_N^1). \end{aligned}$$

We consider $C_0(E^0), C_0(X_k^0)$ as subalgebras of $C_0(E_N^0)$, and $C_d(E^1), C_{d_k}(X_k^1)$ as subspaces of $C_{d_1}(X_1^1)$ for $k = 1, 2, \ldots, N$. We will identify $C_0(X_k^0)$ and $C_{d_k}(X_k^1)$ with $C_0(E^0)$ for each k. Then the map $\theta_{g,g'} \mapsto g\overline{g'}$ for $g, g' \in C_{d_k}(X_k^1) \cong C_0(E^0)$ gives an isomorphism from $\mathcal{K}(C_{d_k}(X_k^1))$ to $C_0(E^0)$. Since $r_k\colon X_k^1 \to X_k^0$ is defined by the identity map of E^0, the map $\pi_{r_k}\colon C_0(X_k^0) \to \mathcal{K}(C_{d_k}(X_k^1))$ is the inverse of the above isomorphism modulo the identification $C_0(X_k^0) \cong C_0(E^0)$. Let us denote by $\{u_{k,l}\}_{0 \le k,l \le N}$ the matrix units of $\mathbb{M}_{N+1}$. We define two maps $T^0\colon C_0(E_N^0) \to \mathcal{O}(E) \otimes \mathbb{M}_{N+1}$ and $T^1\colon C_{d_N}(E_N^1) \to \mathcal{O}(E) \otimes \mathbb{M}_{N+1}$ by

$$T^0(f_0, f_1, \ldots, f_N) = \sum_{k=0}^{N} t^0(f_k) \otimes u_{k,k},$$

$$T^1(\xi, g_1, \ldots, g_N) = t^1(\xi) \otimes u_{0,0} + \sum_{k=1}^{N} t^0(g_k) \otimes u_{k,k-1}.$$

We will show that $T = (T^0, T^1)$ is an injective Cuntz-Krieger E_N-pair. Take $\eta = (\xi, g_1, \ldots, g_N), \eta' = (\xi', g_1', \ldots, g_N') \in C_{d_N}(E_N^1)$. We see that

$$\langle \eta, \eta' \rangle = \big(\langle \xi, \xi' \rangle + \overline{g_1} g_1', \overline{g_2} g_2', \ldots, \overline{g_N} g_N', 0\big) \in C_0(E_N^0).$$

We have

$$\begin{aligned}
&T^1(\eta)^*T^1(\eta') \\
&\quad= \Big(t^1(\xi)\otimes u_{0,0}+\sum_{k=1}^N t^0(g_k)\otimes u_{k,k-1}\Big)^*\Big(t^1(\xi')\otimes u_{0,0}+\sum_{k=1}^N t^0(g'_k)\otimes u_{k,k-1}\Big)\\
&\quad= \Big(t^1(\xi)^*\otimes u_{0,0}+\sum_{k=1}^N t^0(g_k)^*\otimes u_{k-1,k}\Big)\Big(t^1(\xi')\otimes u_{0,0}+\sum_{k=1}^N t^0(g'_k)\otimes u_{k,k-1}\Big)\\
&\quad= \big(t^1(\xi)^*t^1(\xi')\big)\otimes u_{0,0}+\sum_{k=1}^N \big(t^0(g_k)^*t^0(g'_k)\big)\otimes u_{k-1,k-1}\\
&\quad= t^0(\langle\xi,\xi'\rangle)\otimes u_{0,0}+\sum_{k=1}^N t^0(\overline{g_k}g'_k)\otimes u_{k-1,k-1}\\
&\quad= T^0(\langle\eta,\eta'\rangle).
\end{aligned}$$

Take $f=(f_0,f_1,\ldots,f_N)\in C_0(E^0_N)$ and $\eta=(\xi,g_1,\ldots,g_N)\in C_{d_N}(E^1_N)$. We see that $\pi_{r_N}(f)\eta=(\pi_r(f_0)\xi,f_1g_1,\ldots,f_Ng_N)$. We have

$$\begin{aligned}
T^0(f)T^1(\eta) &= \Big(\sum_{k=0}^N t^0(f_k)\otimes u_{k,k}\Big)\Big(t^1(\xi)\otimes u_{0,0}+\sum_{k=1}^N t^0(g_k)\otimes u_{k,k-1}\Big)\\
&= \big(t^0(f_0)t^1(\xi)\big)\otimes u_{0,0}+\sum_{k=1}^N \big(t^0(f_k)t^0(g_k)\big)\otimes u_{k,k-1}\\
&= t^1(\pi_r(f_0)\xi)\otimes u_{0,0}+\sum_{k=1}^N t^0(f_kg_k)\otimes u_{k,k-1}\\
&= T^1(\pi_{r_N}(f)\eta).
\end{aligned}$$

Thus $T=(T^0,T^1)$ is an injective Toeplitz E_N-pair. We will prove that T is a Cuntz-Krieger E_N-pair. Take $f=(f_0,f_1,\ldots,f_N)\in C_0((E^0_N)_{\mathrm{rg}})$. By Lemma 6.1, we have $f_0\in C_0(E^0_{\mathrm{rg}})$. Hence we see that

$$\begin{aligned}
\pi_{r_N}(f_0,f_1,\ldots,f_N) &= \big(\pi_r(f_0),\pi_{r_1}(f_1),\ldots,\pi_{r_N}(f_N)\big)\\
&\in \mathcal{K}(C_d(E^1))\oplus\mathcal{K}(C_{d_1}(X^1_1))\oplus\cdots\oplus\mathcal{K}(C_{d_N}(X^1_N)).
\end{aligned}$$

We compute $\varPhi(\pi_{r_N}(f_0,f_1,\ldots,f_N))\in\mathcal{O}(E)\otimes\mathbb{M}_{N+1}$ where $\varPhi\colon\mathcal{K}(C_{d_N}(E^1_N))\to\mathcal{O}(E)\otimes\mathbb{M}_{N+1}$ is the map induced by the Toeplitz E_N-pair $T=(T^0,T^1)$. For $\xi,\xi'\in C_d(E^1)\subset C_{d_N}(E^1_N)$, we have

$$\varPhi(\theta_{\xi,\xi'})=T^1(\xi)T^1(\xi')^*=(t^1(\xi)\otimes u_{0,0})(t^1(\xi')\otimes u_{0,0})^*=\varphi(\theta_{\xi,\xi'})\otimes u_{0,0}.$$

Hence we get

$$\varPhi(\pi_r(f_0))=\varphi(\pi_r(f_0))\otimes u_{0,0}=t^0(f_0)\otimes u_{0,0}$$

because $f_0\in C_0(E^0_{\mathrm{rg}})$. For $k=1,2,\ldots,N$, we get

$$\varPhi(\pi_{r_k}(f_k))=t^0(f_k)\otimes u_{k,k},$$

by the remark in the beginning of this proof and the computation

$$\begin{aligned}\Phi(\theta_{g_k,g'_k}) &= T^1(g_k)T^1(g'_k)^* \\ &= (t^0(g_k)\otimes u_{k,k-1})(t^0(g'_k)\otimes u_{k,k-1})^* \\ &= t^0(g_k\overline{g'_k})\otimes u_{k,k},\end{aligned}$$

for $g_k, g'_k \in C_{d_k}(X^1_k) \subset C_{d_N}(E^1_N)$. Therefore we have

$$\begin{aligned}\Phi(\pi_{r_N}(f_0,f_1,\ldots,f_N)) &= t^0(f_0)\otimes u_{0,0} + t^0(f_1)\otimes u_{1,1} + \cdots + t^0(f_N)\otimes u_{N,N} \\ &= T^0(f_0,f_1,\ldots,f_N).\end{aligned}$$

This shows that T is a Cuntz-Krieger E_N-pair.

Next we will show that the C^*-algebra $C^*(T)$ generated by the images of T^0 and T^1 is $\mathcal{O}(E)\otimes\mathbb{M}_{N+1}$. Since $T^0(f) = t^0(f)\otimes u_{0,0}$ and $T^1(\xi) = t^1(\xi)\otimes u_{0,0}$ for $f \in C_0(E^0) \subset C_0(E^0_N)$ and $\xi \in C_d(E^1) \subset C_{d_N}(E^1_N)$, we have $\mathcal{O}(E)\otimes u_{0,0} \subset C^*(T)$. For $f \in C_0(X^0_1) \subset C_0(E^0_N)$, we have $T^0(f) = t^0(f)\otimes u_{1,0}$. Hence $t^0(f)x\otimes u_{1,0} \subset C^*(T)$ for all $f \in C_0(E^0)$ and all $x \in \mathcal{O}(E)$. Since we have $t^0(C_0(E^0))\mathcal{O}(E) = \mathcal{O}(E)$ (see [K1, Proposition 2.5] and the remark before it), we get $\mathcal{O}(E)\otimes u_{1,0} \subset C^*(T)$. Recursively, we have $\mathcal{O}(E)\otimes u_{k,0} \subset C^*(T)$ for $k = 2,\ldots,N$. Since $\mathcal{O}(E)\otimes\mathbb{M}_{N+1}$ is generated by $\mathcal{O}(E)\otimes u_{k,0}$ for $k = 0,1,\ldots,N$, we have $C^*(T) = \mathcal{O}(E)\otimes\mathbb{M}_{N+1}$.

Finally we will find a gauge action for the pair T. For each $z \in \mathbb{T}$, define a unitary $u_z \in \mathbb{M}_{N+1}$ by $u_z = \sum_{k=0}^N z^k u_{k,k}$, and an automorphism $\operatorname{Ad} u_z$ on $\mathbb{M}_{N+1}$ by $\operatorname{Ad} u_z(x) = u_z x u_z^*$ for $x \in \mathbb{M}_{N+1}$. Let β be the gauge action on $\mathcal{O}(E)$. The automorphism $\beta'_z = \beta_z \otimes \operatorname{Ad} u_z$ of $\mathcal{O}(E)\otimes\mathbb{M}_{N+1}$ satisfies the equations $\beta'_z(T^0(f)) = T^0(f)$ and $\beta'_z(T^1(\eta)) = zT^1(\eta)$ for $f \in C_0(E^0_N)$ and $\eta \in C_{d_N}(E^1_N)$. Now T gives an isomorphism $\mathcal{O}(E_N) \cong \mathcal{O}(E)\otimes\mathbb{M}_{N+1}$ with the help of Proposition 1.6. □

The discussion above works for $N = \infty$. Namely, if we define a topological graph $E_\infty = (E^0_\infty, E^1_\infty, d_\infty, r_\infty)$ by

$$E^0_\infty = E^0 \amalg X^0_1 \amalg X^0_2 \amalg \cdots, \qquad E^1_\infty = E^1 \amalg X^1_1 \amalg X^1_2 \amalg \cdots \quad (X^0_k, X^1_k \cong E^0),$$

$$\begin{aligned} &d_\infty\colon E^1 \xrightarrow{d} E^0, \quad d_\infty\colon X^1_k \xrightarrow{\mathrm{id}} X^0_{k-1} \qquad &&\text{where } X^0_0 = E^0 \subset E^0_\infty, \\ &r_\infty\colon E^1 \xrightarrow{r} E^0, \quad r_\infty\colon X^1_k \xrightarrow{\mathrm{id}} X^0_k \qquad &&\text{for } k = 1,2,\ldots,\end{aligned}$$

then the proof of Proposition 6.5, with appropriate simple modifications, proves the following proposition.

Proposition 6.6. *We have $\mathcal{O}(E_\infty) \cong \mathcal{O}(E)\otimes\mathbb{K}$.*

Remark 6.7. This Proposition generalizes [T, Theorem 4.2].

Remark 6.8. For a topological graph E and a positive integer N, there are many ways to construct topological graphs E' such that the associated C^*-algebras $\mathcal{O}(E')$ are isomorphic to $\mathcal{O}(E)\otimes\mathbb{M}_{N+1}$. Besides the topological graph E_N defined above, we give another example $\bar{E}_N = (\bar{E}^0_N, \bar{E}^1_N, \bar{d}_N, \bar{r}_N)$. We set $\bar{E}^0_N = E^0_N$, $\bar{E}^1_N = E^1_N$, $\bar{r}_N = r_N$ and define $\bar{d}_N\colon \bar{E}^1_N \to \bar{E}^0_N$ by

$$\bar{d}_N\colon E^1 \xrightarrow{d} E^0, \quad \bar{d}_N\colon X^1_k \xrightarrow{\mathrm{id}} E^0 \quad \text{for } k = 1,2,\ldots,N.$$

We can prove that $\mathcal{O}(\bar{E}_N) \cong \mathcal{O}(E) \otimes \mathbb{M}_{N+1}$ by using the injective Cuntz-Krieger $\bar{E}_N$-pair $\bar{T} = (\bar{T}^0, \bar{T}^1)$ where $\bar{T}^0$ is the same map as T^0 in Proposition 6.5, and

$$\bar{T}^1(\xi, g_1, \ldots, g_N) = t^1(\xi) \otimes u_{0,0} + \sum_{k=1}^{N} t^0(g_k) \otimes u_{k,0},$$

for $(\xi, g_1, \ldots, g_N) \in C_{\bar{d}_N}(\bar{E}_N^1)$, and also using the automorphism $\beta_z \otimes \operatorname{Ad} u'_z$ where $u'_z = u_{0,0} + z\sum_{k=1}^{N} u_{k,k} \in \mathbb{M}_{N+1}$ for $z \in \mathbb{T}$. This construction also works for $N = \infty$.

7. Other operations

Proposition 7.1. *For a topological graph $E = (E^0, E^1, d, r)$, the C^*-algebra $\mathcal{O}(E)$ is unital if and only if E^0 is compact.*

Proof. We have that $\mathcal{O}(E)$ is unital if and only if $t^0(C_0(E^0))$ is unital because the hereditary subalgebra generated by $t^0(C_0(E^0))$ is $\mathcal{O}(E)$ (see [K1, Proposition 2.5]). Since t^0 is injective, $t^0(C_0(E^0))$ is unital if and only if E^0 is compact. We are done. $\square$

Definition 7.2. Let $E = (E^0, E^1, d, r)$ be a topological graph. The topological graph $\widetilde{E} = (\widetilde{E}^0, E^1, d, r)$ is called the one-point compactification of E where $\widetilde{E}^0 = E^0 \cup \{\infty\}$ is the one-point compactification of E^0.

Lemma 7.3. *For the one-point compactification $\widetilde{E}$ of a topological graph $E = (E^0, E^1, d, r)$, we have $\widetilde{E}^0_{\mathrm{rg}} = E^0_{\mathrm{rg}}$.*

Proof. It is clear that $\widetilde{E}^0_{\mathrm{fin}} \cap E^0 = E^0_{\mathrm{fin}}$ and $\widetilde{E}^0_{\mathrm{sce}} \cap E^0 = E^0_{\mathrm{sce}}$. Therefore $\widetilde{E}^0_{\mathrm{rg}} \cap E^0 = E^0_{\mathrm{rg}}$. Since $r^{-1}(\infty) = \emptyset$, we have $\infty \notin \widetilde{E}^0_{\mathrm{rg}}$ by Lemma 1.4. Hence we have $\widetilde{E}^0_{\mathrm{rg}} = E^0_{\mathrm{rg}}$. $\square$

In Lemma 7.3, we see that $\infty \in \widetilde{E}^0_{\mathrm{sg}} = \widetilde{E}^0_{\mathrm{inf}} \cup \overline{\widetilde{E}^0_{\mathrm{sce}}}$. Note that there exist topological graphs E with $\infty \notin \widetilde{E}^0_{\mathrm{inf}}$ (for example, in the case that E^1 is compact) as well as ones with $\infty \notin \overline{\widetilde{E}^0_{\mathrm{sce}}}$ (for example, in the case that $E^1 = E^0$ and $r = \mathrm{id}$).

Proposition 7.4. *Let E be a topological graph, and $\widetilde{E}$ be its one-point compactification. Then $\mathcal{O}(\widetilde{E})$ is isomorphic to the unitization $\mathcal{O}(E)^\sim$ of the C^*-algebra $\mathcal{O}(E)$.*

Proof. Define a $*$-homomorphism $\tilde{t}^0 \colon C(\widetilde{E}^0) \to \mathcal{O}(E)^\sim$ by $\tilde{t}^0(f) = t^0(f - f(\infty)) + f(\infty)$. Then it is easy to see that $\tilde{t} = (\tilde{t}^0, t^1)$ is an injective Toeplitz $\widetilde{E}$-pair which admits a gauge action and satisfies $C^*(\tilde{t}) = \mathcal{O}(E)^\sim$. By Lemma 7.3, the pair $\tilde{t}$ is a Cuntz-Krieger $\widetilde{E}$-pair. Hence by Proposition 1.6, we have $\mathcal{O}(\widetilde{E}) \cong \mathcal{O}(E)^\sim$. $\square$

For a discrete graph $E = (E^0, E^1, d, r)$ with infinitely many vertices, its one-point compactification $\widetilde{E}$ is no longer discrete.

Definition 7.5. We define a *disjoint union* $E \amalg F$ of two topological graphs E and F by $(E \amalg F)^0 = E^0 \amalg F^0$, $(E \amalg F)^1 = E^1 \amalg F^1$ and $d, r \colon (E \amalg F)^1 \to (E \amalg F)^0$ are natural ones. The *disjoint union* $\coprod_{\lambda\in\Lambda} E_\lambda$ of a family of topological graphs $\{E_\lambda\}_{\lambda\in\Lambda}$ is defined similarly.

It is easy to see the following.

Proposition 7.6. *For two topological graphs E and F, we have $\mathcal{O}(E \amalg F) = \mathcal{O}(E) \oplus \mathcal{O}(F)$. We also have $\mathcal{O}(\coprod_{\lambda\in\Lambda} E_\lambda) = \bigoplus_{\lambda\in\Lambda} \mathcal{O}(E_\lambda)$ for a family of topological graphs $\{E_\lambda\}_{\lambda\in\Lambda}$.*

Let $E = (E^0, E^1, d_E, r_E)$ be a topological graph, and X be a locally compact space. We define a topological graph $E \times X$ as follows. We set $(E\times X)^0 = E^0 \times X$, $(E\times X)^1 = E^1 \times X$, and define $d, r\colon (E\times X)^1 \to (E\times X)^0$ by $d((e,x)) = (d_E(e), x)$ and $r((e,x)) = (r_E(e), x)$ for $(e,x) \in (E\times X)^1$. It is easy to see $(E\times X)^0_{\mathrm{rg}} = E^0_{\mathrm{rg}} \times X$.

Proposition 7.7. *We have $\mathcal{O}(E\times X) \cong \mathcal{O}(E) \otimes C_0(X)$.*

Proof. First note that we can identify $C_0((E\times X)^0) = C_0(E^0)\otimes C_0(X)$. We can also see that $C_d((E\times X)^1)$ is isomorphic to the completion of the algebraic tensor product $C_d(E^1)\odot C_0(X)$. We define a $*$-homomorphism $T^0\colon C_0((E\times X)^0) \to \mathcal{O}(E)\otimes C_0(X)$ and a linear map $T^1\colon C_d((E\times X)^1) \to \mathcal{O}(E)\otimes C_0(X)$ by $T^0(f\otimes g) = t^0(f)\otimes g$ and $T^1(\xi\otimes g) = t^1(\xi)\otimes g$ for $f\in C_0(E^0)$, $\xi \in C_d(E^1)$ and $g \in C_0(X)$. It is routine to check that the pair $T = (T^0, T^1)$ is an injective Cuntz-Krieger $E\times X$-pair admitting a gauge action. It is also easy to see that $C^*(T) = \mathcal{O}(E)\otimes C_0(X)$. Hence by Proposition 1.6, we have an isomorphism from $\mathcal{O}(E\times X)$ to $\mathcal{O}(E)\otimes C_0(X)$. □

8. Examples 1

Thanks to the study above, we can show that the class of C^*-algebras arising from topological graphs contains all AF-algebras and many AH-algebras.

Let us take an AF-algebra A and write $A = \varinjlim(A_n, \mu_n)$ where A_n is a finite dimensional C^*-algebra and $\mu_n\colon A_n \to A_{n+1}$ is an injective $*$-homomorphism for $n\in\mathbb{N}$. We write $A_n = \bigoplus_{i=1}^{i_n} A_n^{(i)}$ where $A_n^{(i)} \cong \mathbb{M}_{k_n^{(i)}}$ for a positive integer $k_n^{(i)}$. For each $n\in\mathbb{N}$, a map $\mu_n\colon A_n\to A_{n+1}$ is characterized (up to unitary equivalence) by an $\mathbb{N}$-valued rectangular matrix $(\sigma_n^{(i,j)})_{1\leq j\leq i_n, 1\leq i\leq i_{n+1}}$ where $\sigma_n^{(i,j)}$ is the multiplicity of the map $\mu_n^{(i,j)}\colon A_n^{(j)} \to A_{n+1}^{(i)}$ which is obtained by restricting μ_n. Note that we have $\sum_{j=1}^{i_n} \sigma_n^{(i,j)} k_n^{(j)} \leq k_{n+1}^{(i)}$ for each $n\in\mathbb{N}$ and $1\leq i\leq i_{n+1}$.

For each $n\in\mathbb{N}$, we define a topological graph $E_n = (E_n^0, E_n^1, d_n, r_n)$ as follows:

$$\begin{aligned}
E_n^0 &= \{v_n^{(i,k)} \mid 1\leq i\leq i_n,\ 1\leq k\leq k_n^{(i)}\},\\
E_n^1 &= \{e_n^{(i,k)} \mid 1\leq i\leq i_n,\ 1\leq k\leq k_n^{(i)}-1\},\\
d_n(e_n^{(i,k)}) &= v_n^{(i,k)}, \quad r_n(e_n^{(i,k)}) = v_n^{(i,k+1)}.
\end{aligned}$$

We see that $\mathcal{O}(E_n) \cong \bigoplus_{i=1}^{i_n} \mathbb{M}_{k_n^{(i)}} \cong A_n$ by Proposition 6.5 and Proposition 7.6. We define two maps $m_n^0\colon \widetilde{E}_{n+1}^0 \to \widetilde{E}_n^0$ and $m_n^1\colon \widetilde{E}_{n+1}^1 \to \widetilde{E}_n^1$ as follows. Take $i\in\{1,\ldots,i_{n+1}\}$ and $k\in\{1,\ldots,k_{n+1}^{(i)}\}$. If $k > \sum_{j=1}^{i_n}\sigma_n^{(i,j)}k_n^{(j)}$, then define $m_n^0(v_{n+1}^{(i,k)}) = \infty$ and $m_n^1(e_{n+1}^{(i,k)}) = \infty$. Otherwise, we can find $j\in\{1,\ldots,i_n\}$ such that $k' = k - \sum_{j'=1}^{j-1}\sigma_n^{(i,j')}k_n^{(j')}$ satisfies that $1\leq k'\leq \sigma_n^{(i,j)}k_n^{(j)}$. Take $l\in\{1,\ldots,k_n^{(j)}\}$ with $l\equiv k' \bmod k_n^{(j)}$. We define $m_n^0(v_{n+1}^{(i,k)}) = v_n^{(j,l)}$ and

$$m_n^1(e_{n+1}^{(i,k)}) = \begin{cases} e_n^{(j,l)} & \text{if } 1\leq l\leq k_n^{(j)}-1\\ \infty & \text{if } l = k_n^{(j)}.\end{cases}$$

Then $m_n = (m_n^0, m_n^1)$ is a regular factor map from E_{n+1} to E_n and the $*$-homomorphism $\mathcal{O}(E_n) \to \mathcal{O}(E_{n+1})$ induced by m_n is the same map as the injection μ_n. Hence if we denote by $E = (E^0, E^1, d, r)$ the projective limit of the projective system $\{E_n\}$ and $\{m_n\}$, then we have $A \cong \mathcal{O}(E)$ by Proposition 4.13. Note that E^0 is a totally disconnected space, and $d, r\colon E^1 \to E^0$ are homeomorphisms onto open subsets of E^0. Thus this is an example of a crossed product by partial homeomorphisms explained in Subsection 10.1, and this construction is the same as in [E2].

Note that the class of graph algebras contains all AF-algebras up to strong Morita equivalence, but does not contain many AF-algebras such as simple unital infinite dimensional AF-algebras (e.g. UHF-algebras).

Next we see that many AH-algebras can be obtained as C^*-algebras of topological graphs. For a topological graph $E = (E^0, E^1, d, r)$ with $E^1 = \emptyset$, we have $\mathcal{O}(E) \cong C_0(E^0)$. Thus the class of our algebras contains all commutative C^*-algebras. Combining this fact with Proposition 6.5 and Proposition 7.6, we see that a C^*-algebra A of the form $A = \bigoplus_{k=1}^K C_0(X_k) \otimes \mathbb{M}_{n_k}$, where X_k is a locally compact space and n_k is a positive integer, is obtained as a C^*-algebra $\mathcal{O}(E)$ arising from a topological graph E (we can also use Proposition 7.7).

Let us take C^*-algebras A, B of the form

$$A = \bigoplus_{k=1}^{K} C_0(X_k) \otimes \mathbb{M}_{n_k}, \quad B = \bigoplus_{l=1}^{L} C_0(Y_l) \otimes \mathbb{M}_{m_l},$$

and choose topological graphs E, F such that $\mathcal{O}(E) \cong A$ and $\mathcal{O}(F) \cong B$ as above. Not all $*$-homomorphisms from A to B come from regular factor maps from F to E. However, every diagonal $*$-homomorphism from A to B comes from a regular factor map from F to E, where a $*$-homomorphism $\mu\colon A \to B$ is called diagonal if for each $k \in \{1, \ldots, K\}$ and $l \in \{1, \ldots, L\}$, the restriction $\mu_{l,k}\colon C_0(X_k) \otimes \mathbb{M}_{n_k} \to C_0(Y_l) \otimes \mathbb{M}_{m_l}$ of μ is of the form

$$\mu_{l,k}(f) = \operatorname{diag}\left\{0, \ldots, 0, f \circ m_{l,k}^{(1)}, \ldots, f \circ m_{l,k}^{(\sigma_{l,k})}, 0, \ldots, 0\right\} \in C_0(Y_l) \otimes \mathbb{M}_{m_l}$$

for $f \in C_0(X_k) \otimes \mathbb{M}_{n_k}$ where the $m_{l,k}^{(i)}$ are continuous maps from $\widetilde{Y}_l$ to $\widetilde{X}_k$ preserving ∞ for $i = 1, 2, \ldots, \sigma_{l,k}$. By Proposition 4.13, our class includes all C^*-algebras which are given by inductive limits of C^*-algebras of the form $\bigoplus_{k=1}^K C_0(X_k) \otimes \mathbb{M}_{n_k}$ with diagonal $*$-homomorphisms. In particular, all simple real rank zero AT-algebras and all Goodearl algebras appear as C^*-algebras of topological graphs (see, [Li, Theorem 4.7.5] and [RS, Example 3.1.7]). The C^*-algebras $\mathcal{A}_T$ of totally ordered, compact metrizable sets T defined in [Rø] are also in our class. In particular, the example $\mathcal{A}_{[0,1]}$ of a purely infinite AH-algebra constructed in [Rø] is obtained as the C^*-algebra of a topological graph.

Besides AF-algebras and AH-algebras, many nuclear C^*-algebras satisfying the Universal Coefficient Theorem appear as C^*-algebras of topological graphs, for example purely infinite C^*-algebras (see [K6]) and stabely projectionless C^*-algebras. In [K4], we study the C^*-algebras of topological graphs arising from constant maps in order to analyze the C^*-algebras generated by scaling elements.

9. Examples 2

Our construction of C^*-algebras from topological graphs is motivated by graph algebras. Graph algebras are one of the generalization of Cuntz-Krieger algebras. In this section, we observe that our construction encompasses other generalizations of Cuntz-Krieger algebras.

9.1. **Exel-Laca algebras.** In [EL], R. Exel and M. Laca presented a method for constructing C^*-algebras, now called Exel-Laca algebras, from an infinite matrix with entries in $\{0,1\}$. They introduced these algebras in order to extend the work of Cuntz and Krieger who focused primarily on finite $\{0,1\}$-valued matrices. In [S, Subsection 3.5], Schweizer observed how to present an Exel-Laca algebra as the C^*-algebra of a topological graph.

Remark 9.1. Schweizer [S] called a topological graph a continuous diagram. His presentation of an Exel-Laca algebra, given by a matrix A, in terms of a topological graph was made under the assumption that A has no columns that are identically zero. This assumption can be removed if, in the notation of [S, Subsection 3.5], one adds the characteristic function δ_i to B for each $i \in \mathcal{G}$ such that the i^{th} column of A is identically zero.

9.2. **Matsumoto algebras.** In [M1], K. Matsumoto introduced a method for constructing C^*-algebras, now called *Matsumoto algebras*, from subshifts. When the subshift is a topological Markov shift, then his construction coincides with the construction of Cuntz and Krieger. In [M2], he generalized subshifts by introducing the notion of a λ-graph and he showed how one can attach a C^*-algebra to one of these, generalizing the Matsumoto algebras from [M1] (see [M2, Corollary 4.5]). His construction used topological graphs, which he called continuous graphs. Thus, Matsumoto algebras and λ-graph algebras are all instances C^*-algebras associated to topological graphs.

Remark 9.2. In [CM], an alternate construction of C^*-algebras from subshifts is presented. These are based on λ-graphs and so, ultimately, may be viewed as coming from topological graphs.

10. Examples 3

The class of C^*-algebras of topological graphs contains the ones of graph algebras and of homeomorphism C^*-algebras. Graph algebras generalizes Cuntz-Krieger algebras, and we study two other such classes in the previous section. In this section, we study three classes of C^*-algebras generalizing homeomorphism C^*-algebras.

10.1. **Crossed products by partial homeomorphisms.**

Definition 10.1. Let X be a locally compact space. A *partial homeomorphism* is a homeomorphism σ from an open subset U of X to another open subset V of X.

If a partial homeomorphism $\sigma\colon U \to V$ on X is given, we can define a $*$-homomorphism $\theta\colon C_0(U) \to C_0(V)$ by $\theta(f) = f \circ \sigma^{-1}$. The triple $(\theta, C_0(V), C_0(U))$ is called a *partial automorphism* of $C_0(X)$ in [E1]. R. Exel associated a C^*-algebra $C_0(X) \rtimes_\theta \mathbb{Z}$ with the partial automorphism $(\theta, C_0(V), C_0(U))$ [E1, Definition 3.7].

Instead of giving a definition of the C^*-algebra $C_0(X) \rtimes_\theta \mathbb{Z}$, we give its universal property (see [AEE, Definition 2.4]).

Proposition 10.2 ([AEE, Example 3.2])**.** *The C^*-algebra $C_0(X) \rtimes_\theta \mathbb{Z}$ is generated by the images of a $*$-homomorphism $\rho^0\colon C_0(X) \to C_0(X) \rtimes_\theta \mathbb{Z}$ and a linear map $\rho^1\colon C_0(V) \to C_0(X) \rtimes_\theta \mathbb{Z}$ satisfying*

(i) $\rho^0(f)\rho^1(g) = \rho^1(fg)$,
(ii) $\rho^1(g)\rho^0(f) = \rho^1(\theta(\theta^{-1}(g)f))$,
(iii) $\rho^1(g)\rho^1(h)^* = \rho^0(g\overline{h})$,
(iv) $\rho^1(g)^*\rho^1(h) = \rho^0(\theta^{-1}(\overline{g}h))$ $\qquad (f \in C_0(X),\ g, h \in C_0(V))$.

Moreover $C_0(X) \rtimes_\theta \mathbb{Z}$ is universal among such C^*-algebras.

Remark 10.3. The similar computation done after Definition 1.2 shows that the conditions (i) and (ii) above are automatically satisfied from the conditions (iii) and (iv), respectively.

From a partial homeomorphism $\sigma\colon U \to V$ on X, we can define a topological graph $E = (E^0, E^1, d, r)$ by $E^0 = X$, $E^1 = U$, $r = \sigma$, and d is a natural embedding. We have $C_d(E^1) = C_0(U)$ with the natural inner product and the natural right action. For $f \in C_0(X) = C_0(E^0)$ and $g \in C_0(U) = C_d(E^1)$, we have $\pi_r(f)g = (f \circ \sigma)g \in C_0(U)$.

Lemma 10.4. *We have $E^0_{\mathrm{rg}} = V$, and $\pi_r(f) = \theta_{g,h}$ for $f \in C_0(V)$ where $g, h \in C_0(U)$ satisfy $\theta^{-1}(f) = g\overline{h}$.*

Proof. By Lemma 1.4, E^0_{rg} is the largest open subset of E^0 satisfying the property that the restriction of r to $r^{-1}(E^0_{\mathrm{rg}})$ is a proper surjection onto E^0_{rg}. Hence we have $E^0_{\mathrm{rg}} = V$ because $r\colon E^1 \to E^0$ is a homeomorphism onto $V \subset E^0$.

For $f \in C_0(V)$, we have $\pi_r(f)g' = (f \circ \sigma)g' = \theta^{-1}(f)g'$ for $g' \in C_0(U)$. We also have $\theta_{g,h}g' = g\overline{h}g'$ for $g, h, g' \in C_0(U)$. Now the latter part is easy to see. □

Proposition 10.5. *There exists a natural isomorphism $\mathcal{O}(E) \cong C_0(X) \rtimes_\theta \mathbb{Z}$.*

Proof. From a Cuntz Krieger E-pair $T = (T^0, T^1)$ on a C^*-algebra, we get a $*$-homomorphism $\rho^0 = T^0\colon C_0(X) \to A$ and a linear map $\rho^1 = T^1 \circ \theta^{-1}\colon C_0(V) \to A$. We will show that ρ^0 and ρ^1 satisfies four conditions in Proposition 10.2. By Remark 10.3, it suffices to see (iii) and (iv). Take $f \in C_0(X)$, $g, h \in C_0(V)$. For (iii), we have

$$\begin{aligned}\rho^1(g)\rho^1(h)^* &= T^1(\theta^{-1}(g))T^1(\theta^{-1}(h))^* = \Phi(\theta_{\theta^{-1}(g),\theta^{-1}(h)}) \\ &= \Phi(\pi_r(g\overline{h})) = T^0(g\overline{h}) = \rho^0(g\overline{h}),\end{aligned}$$

by Lemma 10.4. For (iv), we have

$$\rho^1(g)^*\rho^1(h) = T^1(\theta^{-1}(g))^*T^1(\theta^{-1}(h)) = T^0(\theta^{-1}(\overline{g}h)) = \rho^0(\theta^{-1}(\overline{g}h)).$$

We can similarly prove that $T^0 = \rho^0$ and $T^1 = \rho^1 \circ \theta$ define a Cuntz-Krieger E-pair when two maps ρ^0, ρ^1 satisfy four conditions in Proposition 10.2. Hence there exists a natural isomorphism $\mathcal{O}(E) \cong C_0(X) \rtimes_\theta \mathbb{Z}$. □

Remark 10.6. Using Lemma 10.4, Proposition 10.5 follows from [MS, Proposition 2.23].

10.2. **C^*-algebras associated with branched coverings.** In [DM], V. Deaconu and P. S. Muhly defined a C^*-algebra $C^*(X,\sigma)$ from a branched covering $\sigma\colon X \to X$. They define a C^*-correspondence over $C_0(X)$ by taking a completion of $C_c(X \setminus S)$ where S is a branch set of σ. This C^*-correspondence is the same as $C_d(E^1)$ obtained from a topological graph $E = (E^0, E^1, d, r)$ where $E^0 = X$, $E^1 = X \setminus S$, $d = \sigma$, and r is a natural embedding. They showed that $C^*(X,\sigma)$ is isomorphic to the augmented Cuntz-Pimsner algebra of the C^*-correspondence $C_d(E^1)$ over $C_0(E^0)$ ([DM, Theorem 3.2]). Hence by [K1, Proposition 3.9], we see that $C^*(X,\sigma)$ is isomorphic to $\mathcal{O}(E)$.

Remark 10.7. In [DM], a map $\sigma\colon X \to X$ was assumed to be surjective, but the proof of [DM, Theorem 3.2] goes well without assuming this (cf. [DKM, Theorem 7]).

10.3. **C^*-algebras associated with singly generated topological systems.** In [Re2], J. Renault introduces the following notion.

Definition 10.8. A *singly generated dynamical system* (SGDS) is a pair (X,σ) where X is a locally compact topological space and σ is a local homeomorphism from an open subset $\mathrm{dom}(\sigma)$ of X onto an open subset $\mathrm{ran}(\sigma)$ of X.

J. Renault constructed a groupoid $G(X,\sigma)$ from an SGDS (X,σ) by

$$G(X,\sigma) = \{(x, m-n, y) \mid m,n \in \mathbb{N}, x \in \mathrm{dom}(\sigma^m), y \in \mathrm{dom}(\sigma^n), \sigma^m(x) = \sigma^n(y)\}.$$

The groupoid $G(X,\sigma)$ has a topology whose basic open sets are in the form

$$\mathcal{U}(U_0; m_0, m_1; U_1) = \{(x, m_0 - m_1, y) \mid (x,y) \in U_0 \times U_1, \sigma^{m_0}(x) = \sigma^{m_1}(y)\},$$

where U_i is an open subset of $\mathrm{dom}(\sigma^{m_i})$ on which σ^{m_i} is injective for $i = 0,1$. Note that $\mathcal{U}(U_0; m_0, m_1; U_1)$ is homeomorphic to $\sigma^{m_0}(U_0) \cap \sigma^{m_1}(U_1) \subset X$. By this topology, $G(X,\sigma)$ is a locally compact groupoid.

We should remark that in [Re2] a topological space X in an SGDS (X,σ) was not assumed to be locally compact, or even Hausdorff, but eventually X was assumed to be locally compact (hence Hausdorff) and second countable in order to apply the construction in [Re1] to the groupoid $G(X,\sigma)$. We do not assume that X is second countable here because we do not need this assumption and we can apply the construction in [Re1] even though X is not second countable.

In [Re2], J. Renault defined the C^*-algebra $C^*(X,\sigma)$ of an SGDS (X,σ) to be the C^*-algebra of the locally compact groupoid $G(X,\sigma)$. In other words, the C^*-algebra $C^*(X,\sigma)$ is the norm closure of the $*$-algebra $C_c(G(X,\sigma))$ whose operations are defined by

$$fg(x, m-n, y) = \sum_{z,l} f(x, m-l, z) g(z, l-n, y)$$
$$f^*(x, m-n, y) = \overline{f(y, n-m, x)}$$

for $f, g \in C_c(G(X,\sigma))$ with respect to a certain norm (for the detail, see [Re1]).

From an SGDS (X,σ), we have a topological graph $E = (E^0, E^1, d, r)$ by setting $E^0 = X$, $E^1 = \mathrm{dom}(\sigma)$, $d = \sigma$, and r is a natural embedding. Since r is a natural embedding, we have $E^0_{\mathrm{rg}} = \mathrm{dom}(\sigma)$.

Proposition 10.9. *For an SGDS (X,σ), the C^*-algebra $C^*(X,\sigma)$ is naturally isomorphic to $\mathcal{O}(E)$.*

Proof. We can and will identify the open set

$$\{(x,0,x)\in G(X,\sigma)\mid x\in X\}$$

in $G(X,\sigma)$ with X. It is routine to check that the embedding $C_c(X)\to C_c(G(X,\sigma))$ is a $*$-homomorphism. Thus we get an injective $*$-homomorphism $T^0\colon C_0(X)\to C^*(X,\sigma)$. The open set

$$\{(x,1,\sigma(x))\in G(X,\sigma)\mid x\in \operatorname{dom}(\sigma)\}$$

is homeomorphic to $E^1=\operatorname{dom}(\sigma)$, and the embedding $T^1\colon C_c(E^1)\to C_c(G(X,\sigma))$ satisfies $T^1(\xi)^*T^1(\eta)=T^0(\langle\xi,\eta\rangle)$ for $\xi,\eta\in C_c(E^1)$. Thus we get a linear map $T^1\colon C_d(E^1)\to C^*(X,\sigma)$. It is not difficult to see that $T=(T^0,T^1)$ is an injective Toeplitz E-pair. We will show that T is a Cuntz-Krieger E-pair.

Let U be an open subset of $E^1=\operatorname{dom}(\sigma)$ on which $d=\sigma$ is injective. Take $\xi,\eta\in C_c(U)\subset C_d(E^1)$ and set $f=\xi\overline{\eta}\in C_c(U)\subset C_0(E^0)$. We have $\pi_r(f)=\theta_{\xi,\eta}$ in a similar way to the proof of Lemma 10.4. We also have $T^0(f)=T^1(\xi)T^1(\eta)^*$ by straightforward computation. Thus we get $T^0(f)=\Phi(\pi_r(f))$ for all $f\in C_c(U)$ and all $U\subset\operatorname{dom}(\sigma)=E^0_{\mathrm{rg}}$. This shows that $T^0(f)=\Phi(\pi_r(f))$ for all $f\in C_0(E^0_{\mathrm{rg}})$. Thus T is a Cuntz-Krieger E-pair. Hence there exists a $*$-homomorphism $\rho\colon\mathcal{O}(E)\to C^*(X,\sigma)$. Since the cocycle $G(X,\sigma)\ni(x,k,y)\mapsto k\in\mathbb{Z}$ gives an action $\mathbb{T}\curvearrowright C^*(X,\sigma)$ which is a gauge action of T, the map ρ is injective by Proposition 1.6.

The proof ends once we show that ρ is surjective. To do so, it suffices to see that for all $(x_0,k,x_1)\in G(X,\sigma)$, there exists a neighborhood W of (x_0,k,x_1) such that $C_c(W)\subset C^*(X,\sigma)$ is in the image of ρ. Take $(x_0,k,x_1)\in G(X,\sigma)$. Then there exist $m_0,m_1\in\mathbb{N}$ such that $m_0-m_1=k$, $x_i\in\operatorname{dom}(\sigma^{m_i})$ for $i=0,1$ and $\sigma^{m_0}(x_0)=\sigma^{m_1}(x_1)$. For each $i=0,1$, take a neighborhood $U_i\subset\operatorname{dom}(\sigma^{m_i})$ of x_i on which σ^{m_i} is injective. Set $W=\mathcal{U}(U_0;m_0,m_1;U_1)$ which is a neighborhood of $(x_0,k,x_1)\in G(X,\sigma)$. Let us set $W'=\sigma^{m_0}(U_0)\cap\sigma^{m_1}(U_1)$. Then $W\ni(x,k,y)\mapsto\sigma^{m_0}(x)\in W'$ is a homeomorphism. Take $f\in C_c(W)$. We have $f'\in C_c(W')$ such that $f'(\sigma^{m_0}(x))=f((x,k,y))$ for all $(x,k,y)\in W$. Let X be the support of f'. There exist $X_0\subset U_0$ and $Y_0\subset U_1$ such that $\sigma^{m_0}(X_0)=\sigma^{m_1}(Y_0)=X$. We set $X_n=\sigma^n(X_0)$ for $n=1,\ldots,m_0-1$, and $Y_n=\sigma^n(Y_0)$ for $n=1,\ldots,m_1-1$. For $n=0,\ldots,m_0-1$, choose $\xi_n\in C_c(\operatorname{dom}(\sigma))$ so that $\xi_n(x)=1$ for $x\in X_n$. Similarly for each $n=0,\ldots,m_1-1$, we choose $\eta_n\in C_c(\operatorname{dom}(\sigma))$ so that $\eta_n(x)=1$ for $x\in Y_n$. Then it is not difficult to check

$$f=T^1(\xi_0)\cdots T^1(\xi_{m_0-1})T^0(f')T^1(\eta_{m_1-1})^*\cdots T^1(\eta_0)^*\in\rho(\mathcal{O}(E)).$$

Thus $C_c(W)\subset\rho(\mathcal{O}(E))$. This completes the proof. □

By Proposition 10.9, all C^*-algebras of SGDS's are obtained as C^*-algebras of topological graphs. Conversely we will see in [K7] that from a topological graph we can construct an SGDS so that they define the same C^*-algebra. Thus the class of C^*-algebras of topological graphs coincides with the one of SGDS's.

Proposition 10.10. *The SGDS (X,σ) is essentially free if and only if the topological graph E is topologically free.*

Proof. Since every vertices of E receives at most one edge, every loop has no entrances. Thus E is topologically free if and only if the set of base points of loops has an empty interior. By Baire's theorem, this is equivalent to say that for every positive integer n the set of base points of loops with length n has an empty interior (see [K5, Proposition 6.10] for the detail). The point $x \in E^0 = X$ is a base point of a loop with length n if and only if $x \in \mathrm{dom}(\sigma^n)$ and $\sigma^n(x) = x$. Thus we have shown that the topological graph E is topologically free if and only if the set

$$\{x \in \mathrm{dom}(\sigma^n) \mid \sigma^n(x) = x\}$$

has an empty interior for all positive integer n. This is equivalent to the essential freeness of the SGDS (X, σ) defined in [Re2, Definition 2.5] □

When a local homeomorphism $\sigma\colon \mathrm{dom}(\sigma) \to \mathrm{ran}(\sigma)$ is a partial homeomorphism, the C^*-algebra $C^*(X, \sigma)$ is naturally isomorphic to the C^*-algebra considered in Subsection 10.1, and when σ is obtained by restricting a branched covering $\sigma\colon X \to X$ to the nonsingular set, $C^*(X, \sigma)$ is naturally isomorphic to the C^*-algebra considered in Subsection 10.2.

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Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, 060-0810, JAPAN

E-mail address: katsura@math.sci.hokudai.ac.jp